Reconstruction Schemes of f(Q) Gravity in Accelerating Universe

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DECLARATION BY THE STUDENT

I hereby declare that the work which is being presented in this dissertation entitled *"Reconstruction Schemes of f(Q) Modified Gravity Models in Accelerating Universe"*. In partial fulfilment of the requirements for the award of the degree of Bachelors/Masters/Ph.D of **Master of Science in Applied Mathematics** is an authentic record of my own work carried out during the period from **6/01/2025** to **05/05/2025** under the supervision of **Dr.Sayani Maity Roy** Department of Mathematics, SNU.

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1 Introduction

1.1 Introduction

What is Universe? How was it created? How vast is it? How old is the universe? Is the universe expanding? What is the ultimate fate of our universe? How did galaxies form? Since time immemorial, these fundamental questions have formed an integral part of human thought. Cosmology deals with these questions. That is why no branch of science can have a bigger area of interest than cosmology, the study of the universe in which we live. There are hundreds of billions of galaxies in the universe. A galaxy is a massive, gravitational bound system that consists of stars and stellar remnants, an interstellar medium of gas dust, and an important but poorly understood component tentatively dubbed dark matter. Cosmology is mainly concerned with the extragalactic world. It is a study of the large scale structure of the universe extending to distances of billions of light years – a study of the overall dynamical and physical behavior of billions of galaxies spread across vast distances and of the evolution of this enormous system over several billions years. Galaxies are the largest single aggregates of stars in the universe. A super cluster may be ten times as massive, extending upto 50 mpc in diameter. The most abundant element in the universe is hydrogen, which makes up about 3/4 of all matter!

In the early 1920s, Based on his measurements Edwin Hubble made the discov-

ery that all of the galaxies were moving away from us and that the recession velocity followed a straightforward formula known as Hubble's law: $v = H_0 d$, where v is the recession velocity, d is the distance, and H_0 is the Hubble constant. Hubble's law indicates that the universe is facing an accelerated expansion. In 1998, the observational data of high red-shift Supernova experiments, compiled by two different teams of astrophysicists, presented the direct proof of cosmic acceleration [1, 3]. Indirect evidence comes from the Cosmic Microwave Background (CMB) [9] fluctuations and through large scale structure of the universe. Recent cosmological observational data analysis from WMAP [67, 6] also points to an accelerating universe. The search for theoretical explanations of this feature was assured by the observational evidence for the current expansion of the universe that came from these several resources. The existence of an exotic kind of fluid with negative pressure known as "dark energy" is one theory that could explain this occurrence. Its source has not yet been found, despite years of intense research. Dark energy is predominant at this point in the universe's evolution because it is different from regular matter like baryons and radiation. According to current cosmological observational data, this dark energy accounts for 70 percent of the universe's total energy density.

1.2 Einstein Solution

Newtonian Gravity describes by Poisson's equation:

$$\nabla^2 \phi = 4\pi G \rho \tag{1.1}$$

where ϕ is known as gravitational potential and ρ is known as mass density. Here Einstein points out that Newtonian gravity has uniform mass density. Since all points in the universe are basically equivalent, Einstein suggested [48]:

$$\nabla^2 \phi - \Lambda \phi = 4\pi G \rho \tag{1.2}$$

$$\Rightarrow \qquad 0 - \Lambda \phi = 4\pi G \rho \tag{1.3}$$

At this point

$$\phi_{avg} = \frac{4\pi G}{\Lambda}\rho_{avg} \tag{1.4}$$

and General Relativity

$$G_{\mu v} = \frac{8\pi G}{c^4}T_{\mu v} \tag{1.5}$$

where gravitational potential and the mass density ρ are replaced by the Einstein Tensor and the energy momentum tensor respectively. But Einstein has some objections. Instead of infinite flat universe, Einstein proposed that the universe is spatially finite and has a positive curvature. This allows for a spherical universe which is not static that is which can expand or contract overtime.

$$G\mu v - \Lambda g_{\mu v} = \frac{8\pi G}{c^4}T_{\mu v} \tag{1.6}$$

The term Λ is required for a static universe with positive curvature where Λ is known as Cosmological Constant which is determined by the average mass density of the universe

$$\Lambda = \frac{1}{R^2}, \tag{1.7}$$

where R is known as the radius of the universe. Einstein proposed that the universe is spherical and static

$$G_{\mu v} - \Lambda g_{\mu v} = \frac{8\pi G}{c^4}T_{\mu v}. \tag{1.8}$$

Einstein believed that adding this additional term to this equation was acceptable because it did not violate any laws of conservation of energy momentum:

$$\nabla_{\mu}G^{\mu v} = 0, \tag{1.9}$$

$$\nabla_{\mu}T^{\mu v} = 0, \tag{1.10}$$

$$\nabla_{\mu}g^{\mu v} = 0. \tag{1.11}$$

Einstein assumed $\Lambda << 1$ and most GTR applications use $\Lambda = 0$. Einstein concluded that the spatial part of the universe is positively curved and consequently it requires the newterm $-\Lambda g_{\mu v}$ to model a static universe.

Alexender Friedmann in the year 1922 published the 2 Friedmann equation based on the curvature of space and describes the evolution of the universe overtime according to general relativity.

1.3 Modern Overview of Friedmann equations:

At large scale,universe is the same at all spatial points,universe has uniform mass density ρ (everywhere) and the universe can have a time varying scale factor $a(t)$ as the metric tensor.All spatial points in the universe are on equal footing at large scales but there was no strong evidence to suggest the universe was the same over large scales of time.In GTR this is what we mean by "The universe is expanding. Friedmann equations assumptions states that the universe has time varying scale factor $a(t)$ and the universe has uniform mass density ρ and pressure p when we put this assumption on:

$$G_{\mu v} - \Lambda g_{\mu v} = \frac{8\pi G}{c^4} T_{\mu v} \tag{1.12}$$

We can derive two eqn:

$$\frac{\dot{a} + kc^2}{a^2} = \frac{8\pi G\rho + \Lambda c^2}{3} \tag{1.13}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\Big(\rho + \frac{3p}{c^2}\Big) + \frac{\Lambda c^2}{3} \tag{1.14}$$

This two equations are known as universe expansion rate and universe acceleration rate respectively.

For example if we draw a bunch of dots on the balloon then blow it up all the points will move away from each other as the balloon expands, we are not adding the more balloon material here.It's just as the fabric of the balloon material is being stretched causing points on the balloon to separate.This is exactly the expansion of the universe is explained in GTR.

Friedmann equation,static solution we assume $\dot{a} = \ddot{a} = 0$

$$\Lambda = \frac{4\pi G}{3}\Big(\rho + \frac{3p}{c^2}\Big) \tag{1.15}$$

Actual astronomical evidence for an expanding universe i.e, Doppler effect if we have a stationary source that circulating waves in all directions the waves on all slides of the source will have the same wavelength. But if the source are moving the wave-fronts will bunch up and give them a higher frequency, and the wave-fronts lacking behind the source will spread out giving them the lower frequency.

Inside an atom we know that electrons exist around an atom of the nucleus on only very specific orbitals.When an electron drop from a high energy orbital,the energy difference is released as light at a very specific frequency and this frequency is exactly corresponds to the energy given up by the electron as given by the Planck relation $E = hf$ where E is known as Energy h is Planck constant and f is termed as Frequency. Conversely if an atom absorbs light they can only absorbs light at a very specific frequency that correspond to an electron jumping from a lower energy orbital to a higher energy orbital.
Lights from stars/galaxies will be missing specific frequencies due to absorption by elements present in atmospheric gasses.Standard Hydrogen absorption spectrum we can tribute this shift is either blue-shift or redshift from the doppler effect [54].

In 1912 Vesto Slifur first observed light from distant galaxies was red-shifted.Before Einstein completed GTR(1915).

In 1927 George Lemaitre built on Einstein paper and observation of galaxies red-shifted and was proposed that the universe was expanding.

All spatial points are equivalent and the universe has no center.A homogeneous radius accounting for the radial velocity of extra galactic nebulae and was proposed the universe was expanding and the universe can have the constant positive curvature. And the radial velocity is proportional to the radius of the earth r,a fact which is known as Hubble law.

George Lemaitre states that in the distant past, the universe expanded from a single point. And called the "The Primeval Atom" and same idea as "The Big Bang Theory".

In 1929 Edwin Hubble studied redshift of distant galaxies and wad proposed linear relationship between galaxies $v = Hr$ where r is the radius from earth r and its radial velocity v $\frac{v}{r} = H = \frac{\dot{a}}{a}$ from the universe of scale factor $a(t)$ where H is known as Hubble parameter. Einstein after learning Hubble work, abandoned cosmological constant $\Lambda = 0$.

George Gamow reported that Einstein said Λ was his "Biggest Blunder".

- Cosmic Microwave Background (CMB) Radiation:

Here low-energy electromagnetic radiation left over from the early universe.It also exist everywhere in space and travel in all directions.

In 1948 Ralph Alpher published a paper on the formation of heavy elements in the early universe. Basically on the early universe dominated by matter will have allowed for having elements to form a much greater quantities than we see today. But if the early universe was dominated by high energy radiation the large qaunatity of high energy photon will not electron to loose any time to become bound to a proton preventing any atom from forming.

Universe after radiation has "cooled" is known as "Recombination" where phase in universe history when radiation has cooled enough to allow atoms to form.

If Big-Bang theory is correct then the early universe radiation should still be around. And will have much larger wavelength/lower frequency due to the expansion of the uni-

verse. CMB radiation is NOT predicted for an eternal, steady state universe.

In 1964, CMB detected by Holographic model Horn Antenna, when aimed at empty space, detects a near uniform noise signals and where frequencies distribution matched a black body of 2.725k.

Planck's law for blackbody radiation at temperature T:

$$B(\lambda, T) = \frac{2hc^5}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda \kappa_B T}} - 1} \tag{1.16}$$

Universe is expanding and photons wavelength are increasing and loosing its energy. So the universe is cooling overtime. Low temperature radiation permitting all regions of the universe in the present day is exactly what we would expect to see if the early universe originally was smaller and hotter.

The discovery of the Λ may be nonzero, and the Hubble publication on the galaxy for the redshift was established in 1929 and was initially assumed as $\Lambda = 0$.

In the year 1990 all the information was collected and determined that the universe is accelerating which gives us simplest explanation:$\Lambda \geq 0$.

In 1998 there are two independent groups which was observed by supernova from the galaxies at different distance from earth.Light from farther away is light from earlier in the universe's history.

Hence we concluded that speeds from various galaxies traveling away from the Milky way are increasing overtime, and the universe is expanding and accelerating.

Friedmann equations:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\Big(\rho + \frac{3p}{c^2}\Big) + \frac{\Lambda c^2}{3} \tag{1.17}$$

Here cosmological constant is known as Dark Energy.It is a type of energy responsible for the accelerating and expansion of our universe.

Standard form can be written as:

$$R_{\mu v} - \frac{1}{2}Rg_{\mu v} - \Lambda g_{\mu v} = \frac{8\pi G}{c^4}T_{\mu v} \tag{1.18}$$

Trace Reversed form can be written as:

$$R_{\mu v} = \frac{8\pi G}{c^4}\Big(T_{\mu v} - \frac{1}{2}Tg_{\mu v}\Big) - \Lambda g_{\mu v} \tag{1.19}$$

1.4 Friedmann Lemaitre Robertson Walker(FLRW)

The type of metric which express the expansion of our universe with respect to time and used in a modern cosmology.

- FLRW metric: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{(a(t))^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & -(a(t)r)^2 & 0 \\ 0 & 0 & 0 & -(a(t)rsin\theta)^2 \end{bmatrix}$

It depends on the value of k where $k = 1, 0, -1$ for closed, flat and open universe respectively. Now we will derive the metric expression that elaborate the large scale geometry for the entire universe. This metric is known as FLRW metric.

The metric also can be elaborated from the Fundamental Assumption which we called as Cosmological Principal.

(1)Homogeneous where the universe will look same from all points in the space,it dosen't have any kind of spatial translations and the universe does not have any spatial center.

(2)Isotropic where the universe will look same in all directions and does not change under any spatial rotations.

3 possible spatially homogeneous and isotropic universe is closed,flat and open. Technically there are the geometries that are spatially homogeneous and isotropic. Here we

will consider "Simply connected" geometries(no holes).

We will focus on 1st Flat space case:

Flat(Minkowski) metric,Cartesian coordinates:-

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Flat(Minkowski) metric,Spherical coordinates:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2(sin\theta)^2 \end{bmatrix}$$

Hence the metric is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -(a(t))^2 & 0 & 0 \\ 0 & 0 & -(a(t)r)^2 & 0 \\ 0 & 0 & 0 & -(a(t)rsin\theta)^2 \end{bmatrix}$$

The metric represent a flat space and changes over time, and according to the scale factor $a(t)$ taking as a hole metric represent a curved spacetime which was described by the non-zero Riemann curvature tensor.

Now we shall consider the closed/spherical universe. In 2-dimensional sphere surface all points are equivalent and all directions are same. But our universe spatial universe in 3D:

Equation of sphere:

$$x^2 + y^2 + z^2 = R^2 \tag{1.20}$$

where, $x = Rcos\theta$, $y = Rsin\theta cos\phi$, $z = Rsin\theta sin\phi$

$$\Big(Rcos\theta\Big)^2 + \Big(Rsin\theta cos\phi\Big)^2 + \Big(Rsin\theta sin\phi\Big)^2$$

$$= R^2\Big[\Big(cos\theta\Big)^2 + \Big(sin\theta\Big)^2\Big(cos\phi\Big)^2 + \Big(sin\phi\Big)^2\Big]$$

$$= R^2\Big[\Big(\cos\theta\Big)^2 + \Big(\sin\theta\Big)^2\Big] = R^2 = 1$$

Equation of a 3-sphere:

where $x = R\cos\theta$ $y = R\sin\theta\cos\phi$,

$z = Rsin\theta sin\phi\cos\chi$ $w = Rsin\theta\sin\phi sin\chi$

$$= \Big(Rcos\Big)^2 + \Big(Rsin\theta cos\phi\Big)^2 + \Big(R\sin\theta\cos\phi\cos\chi\Big)^2 + \Big(R\sin\theta\sin\phi\sin\chi\Big)^2$$

$$= R^2\Big[\Big(\cos\theta\Big)^2 + \Big(\sin\theta\Big)^2\Big(cos\phi\Big)^2 + \Big(sin\phi\Big)^2\Big] = R^2 = 1$$

We will consider 3D sphere in our universe in 3-dimensions:

$$\frac{\partial}{\partial\theta} = -R\sin\theta\frac{\partial}{\partial x} + Rcos\theta cos\phi\frac{\partial}{\partial y} + Rcos\theta sin\phi cos\chi\frac{\partial}{\partial z} + Rcos\theta sin\phi sin\chi\frac{\partial}{\partial w} \quad (1.21)$$

$$\frac{\partial}{\partial\phi} = 0\frac{\partial}{\partial x} - Rsin\theta sin\phi\frac{\partial}{\partial y} + Rsin\theta cos\phi cos\chi\frac{\partial}{\partial z} + Rsin\theta cos\phi sin\chi\frac{\partial}{\partial w} \quad (1.22)$$

$$\Rightarrow \qquad \frac{\partial}{\partial\chi} = 0\frac{\partial}{\partial x} + 0\frac{\partial}{\partial y} - R\sin\theta\sin\phi\sin\chi\frac{\partial}{\partial z} + R\sin\theta\cos\phi\cos\chi\frac{\partial}{\partial w} \quad (1.23)$$

We can use two system 1st is $xyzw$ space basis which is commonly known as orthonormal and $\theta\phi\chi$ space basis.And off diagonal metric are $g_{\theta\phi} = g_{\phi\chi} = g_{\theta\chi} = 0$

$$\frac{\partial}{\partial\theta}\frac{\partial}{\partial\theta} = g_{\theta\theta} = R^2 \quad (1.24)$$

$$\frac{\partial}{\partial\phi}\frac{\partial}{\partial\phi} = g_{\phi\phi} = R^2\Big(\sin\theta\Big)^2 \quad (1.25)$$

$$\frac{\partial}{\partial\chi}\frac{\partial}{\partial\chi} = g_{\chi\chi} = R^2\Big(\sin\theta\Big)^2\Big(\sin\phi\Big)^2 \quad (1.26)$$

where the 3-metric space;

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & (\sin\theta)^2 & 0 \\ 0 & 0 & (\sin\theta)^2(sin\phi)^2 \end{bmatrix}$$

Now Relabel coordinates $\theta \to \chi$, $\phi \to \theta$, $\chi \to \phi$

$$\frac{\partial}{\partial\chi}\frac{\partial}{\partial\chi} = g_{\chi\chi} = R^2 \quad (1.27)$$

$$\frac{\partial}{\partial\theta}\frac{\partial}{\partial\theta} = g_{\theta\theta} = R^2\Big(sin\chi\Big)^2 \quad (1.28)$$

$$\frac{\partial}{\partial\phi}\frac{\partial}{\partial\phi} = g_{\phi\phi} = R^2\Big(sin\chi\Big)^2\Big(sin\theta\Big)^2 \tag{1.29}$$

3D flat metric space in spherical polar coordinates:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2(\sin\theta)^2 \end{bmatrix}$$

3 sphere metric in hyper spherical coordinates $R = 1$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & (\sin\chi)^2 & 0 \\ 0 & 0 & (\sin\chi)^2(sin\theta)^2 \end{bmatrix}$$

Spatially Homogeneous, Isotropic metric for 3-D positive curvature:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & (\sin\chi)^2 & 0 \\ 0 & 0 & (\sin\chi)^2(sin\theta)^2 \end{bmatrix}$$

Spatially Homogeneous, Isotropic metric for 3-D zero curvature:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2(\sin\theta)^2 \end{bmatrix}$$

Spatially Homogeneous, Isotropic metric for 3-D negative curvature:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & (\sin\chi)^2 & 0 \\ 0 & 0 & (\sin\chi)^2(\sin\theta)^2 \end{bmatrix}$$

Now there are 4 models we can identified:

(i)Hyperboloid Model are a type of geodesic that are forward walking paths constrained on hyperboloid.

(ii)Poincare Model are a type of geodesic that are circular area and diameter lines ends up like a disc.

(iii)Klein Disc Model are a type of geodesic that are straight lines on a disc.

(iv)Poincare Half Plane Model are a type of geodesic which gives us the vertical and semi circular areas that are centered on a straight line.

Now we know that,

$q = Rcosh\chi, \qquad x = Rsinh\chi cos\theta, \qquad y = Rsinh\chi sin\theta$

$$x^2 + y^2 - q^2 = -R^2 \tag{1.30}$$

$$= \Big(Rsinh\chi cos\theta\Big)^2 + \Big(yRsinh\chi sin\theta\Big)^2 - \Big(Rcosh\chi\Big)^2$$

$$= R^2\Big[\Big(sin\chi\Big)^2\Big(cos\theta\Big)^2 + \Big(sinh\chi\Big)^2\Big(sin\theta\Big)^2 - \Big(cosh\chi\Big)^2\Big]$$

$$= R^2\Big[\Big(\sin h\chi\Big)^2 - \Big(\cosh\chi\Big)^2\Big] = -R^2$$

Now we know that Hyperbolic is essentially same as the Euclidean line:

$q = Rcosh\chi, \qquad x = Rsinh\chi cos\theta, \qquad y = Rsinh\chi sin\theta, \qquad z = Rsinh\chi sin\theta sin\phi$

$$-q^2 + x^2 + y^2 + z^2 \tag{1.31}$$

$$= \Big[\Big(-R\cosh\chi\Big)^2 + \Big(R\sinh\chi\cos\theta\Big)^2 + \Big(R\sinh\chi\sin\theta\Big)^2 + \Big(R\sinh\chi\sin\theta\sin\phi\Big)^2\Big]$$

$$= R^2\Big[-\Big(\cosh\chi\Big)^2 + \Big(\sinh\chi\Big)^2\Big(\cos\theta\Big)^2 + \Big(\sinh\chi\Big)^2\Big(\sin\theta\Big)^2\Big]$$

$$= R^2\Big[-\Big(\cosh\chi\Big)^2 + \Big(\sinh\chi\Big)^2\Big] = -R^2$$

We are only interested in hyperbolic 3-space:

$$\frac{\partial}{\partial\chi} = \frac{\partial q}{\partial\chi}\frac{\partial}{\partial q} + \frac{\partial x}{\partial\chi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\chi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial\chi}\frac{\partial}{\partial z} \tag{1.32}$$

$$\frac{\partial}{\partial\theta} = \frac{\partial q}{\partial\theta}\frac{\partial}{\partial q} + \frac{\partial x}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\theta}\frac{\partial}{\partial y} + \frac{\partial z}{\partial\theta}\frac{\partial}{\partial z} \tag{1.33}$$

$$\frac{\partial}{\partial\phi} = \frac{\partial q}{\partial\phi}\frac{\partial}{\partial q} + \frac{\partial x}{\partial\phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\phi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial\phi} + \frac{\partial}{\partial z} \tag{1.34}$$

Therefore,

$$\frac{\partial}{\partial\chi} = R\Big[\sinh\chi\frac{\partial}{\partial q} + \cosh\chi\cos\theta\frac{\partial}{\partial x} + \cosh\chi\sin\theta\cos\phi\frac{\partial}{\partial y} + \theta\sin\phi\frac{\partial}{\partial z}\Big] \tag{1.35}$$

$$\frac{\partial}{\partial\theta} = R\Big[0\frac{\partial}{\partial q} - \sinh\chi\sin\theta\frac{\partial}{\partial x} + \sinh\chi\cos\theta\cos\phi\frac{\partial}{\partial y} + \sinh\chi\cos\theta\sin\phi\frac{\partial}{\partial z}\Big] \tag{1.36}$$

$$\frac{\partial}{\partial\phi} = R\Big[0\frac{\partial}{\partial q} + 0\frac{\partial}{\partial x} - \sinh\chi\sin\theta\sin\phi\frac{\partial}{\partial y} + \sinh\chi\sin\theta\cos\phi\frac{\partial}{\partial z}\Big] \tag{1.37}$$

We know that $qxyz$ is a space metric which is demonstrated by:

$$-q^2 + x^2 + y^2 + z^2 = -R^2 \tag{1.38}$$

$$\frac{\partial}{\partial q}\frac{\partial}{\partial q} = g_{qq} = -1 \tag{1.39}$$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial x} = g_{xx} = 1 \tag{1.40}$$

$$\frac{\partial}{\partial y}\frac{\partial}{\partial y} = g_{yy} = 1 \tag{1.41}$$

$$\frac{\partial}{\partial z}\frac{\partial}{\partial z} = g_{zz} = 1 \tag{1.42}$$

Spherical closed:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & (\sin\chi)^2 & 0 \\ 0 & 0 & (\sin\chi)^2(sin\theta)^2 \end{bmatrix}$$

Flat:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2(\sin\theta)^2 \end{bmatrix}$$

Hyperbolic open:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & (\sin\chi)^2 & 0 \\ 0 & 0 & (\sin\chi)^2(\sin\theta)^2 \end{bmatrix}$$

Closed:$g_{\mu v}$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -(a(t))^2 & 0 & 0 \\ 0 & 0 & -(a(t)\sin\chi)^2 & 0 \\ 0 & 0 & 0 & -(a(t)\sin\chi\sin\theta)^2 \end{bmatrix}$$

Flat for $g_{\mu v}$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -(a(t))^2 & 0 & 0 \\ 0 & 0 & -(a(t)\chi)^2 & 0 \\ 0 & 0 & 0 & -(a(t)\chi\ \sin\theta)^2 \end{bmatrix}$$

FLRW metric $g_{\mu v}$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{(a(t))^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & -(a(t)r)^2 & 0 \\ 0 & 0 & 0 & -(a(t)r\sin\theta)^2 \end{bmatrix}$$

Here hyper spherical coordinates $\chi\theta\phi$ are circular angles $r = sin\chi$ are reduced to circumference polar coordinates $r\theta\phi$ as $r = sin\chi$

FLRW Tensors :

$g_{\mu v}$ are all 4 spacetime indices and the metric of the tensors will be denoted as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^2\gamma_{11} & 0 & 0 \\ 0 & 0 & -a^2\gamma_{22} & 0 \\ 0 & 0 & 0 & -a^2\gamma_{33} \end{bmatrix}$$

Hence, $g^{\mu v}$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{-1-kr^2}{a^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{a^2r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{a^2r^2(\sin\theta)^2} \end{bmatrix}$$

Given $g_{00} = 1, \qquad g_{11} = -\frac{1}{1-kr^2}, \qquad g_{22} = -\frac{1}{a^2r^2}, \qquad g_{33} = -a^2r^2(sin\theta)^2.$

$g^{00} = 1, \qquad g^{11} = -\frac{1-kr^2}{a^2}, \qquad g^{22} = -\frac{1}{a^2r^2}, \qquad g^{33} = -\frac{1}{a^2r^2(sin\theta)^2}$

$$\Gamma^{\sigma}_{\mu v} = \frac{1}{2}g^{\sigma\alpha}\Big(\partial_v g_{\alpha v} + \partial_\mu g_{\sigma v} - \partial_\alpha g_{\mu v}\Big) \tag{1.43}$$

$$\Gamma^{\sigma}_{\mu v} = \frac{1}{2}g^{\sigma\sigma}\Big(\partial_v g_{\sigma\mu} + \partial_\mu g_{\sigma v} - \partial_\sigma g_{\mu v}\Big) \tag{1.44}$$

$$\Gamma^{0}_{\mu v} = \frac{1}{2}g^{00}\Big(\partial_v g_{0\mu} + \partial_\mu g_{0v} - \partial_0 g_{\mu v}\Big) \tag{1.45}$$

$$\Gamma^0_{00} = \frac{1}{2}g^{00}\Big(\partial_0 g_{00}\Big) = 0$$

$$\Gamma^0_{ii} = \frac{1}{2}\Big(\partial_i g_{0i} + \partial_i g_{0i} - \partial_0 g_{ii}\Big) \tag{1.46}$$

$$= \frac{1}{2}\Big(- \partial_0 g_{ii}\Big)$$

$$= \frac{1}{2c}\frac{\partial}{\partial t}\Big(- a^2\gamma_{ii}\Big) = -\frac{a\dot{a}}{c}\gamma_{ii}$$

$$\Gamma^0_{11} = -\frac{a\dot{a}}{c}\Big(- \frac{1}{1-kr^2}\Big)$$

$$\Gamma^0_{22} = -\frac{a\dot{a}}{c}\Big(- r^2\Big)$$

$$\Gamma^0_{33} = -\frac{a\dot{a}}{c}\Big(- r^2\Big(sin\theta\Big)\Big)$$

$$\Gamma^1_{\mu v} = \frac{1}{2}\Big(\partial_v g_{1v} + \partial_\mu g_{1v} - \partial_1 g_{\mu v}\Big) \tag{1.47}$$

$$\Gamma^1_{00} = \frac{1}{2}g^{11}\Big(\partial_1 g_{01} + \partial_1 g_{10} - \partial_1 g_{00}\Big) = \frac{1}{2}g^{11}\Big(\partial_1 g_{00}\Big) = 0$$

$$\Gamma^1_{11} = \frac{1}{2}\Big(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}\Big) = \frac{1}{2}g^{11}\Big(\partial_1 g_{11}\Big)$$

$$= \frac{1}{2}\Big(- \frac{1-kr^2}{a^2}\Big)\partial_r\Big(- \frac{a^2}{1-kr^2}\Big) = \frac{kr}{1-kr^2}$$

$$\Gamma^1_{22} = \frac{1}{2}g^{11}\Big(\partial_1 g_{12} + \partial_1 g_{12} - \partial_1 g_{22}\Big) = \frac{1}{2}g^{11}\Big(- \partial_1 g_{12}\Big)$$

$$= \frac{1}{2}\Big(- \frac{1-kr^2}{a^2}\Big)\partial_r\Big(a^2 r^2\Big) = -\Big(1-kr^2\Big)r$$

$$\Gamma^1_{23} = \frac{1}{2}g^{11}\Big(\partial_1 g_{13} + \partial_1 g_{13} - \partial_1 g_{33}\Big) = \frac{1}{2}g^{11}\Big(- \partial_1 g_{33}\Big)$$

$$= \frac{1}{2}\Big(- \frac{1-kr^2}{a^2}\Big)\partial_r\Big(- a^2 r^2\Big(\sin\theta\Big)\Big)$$

$$= \frac{1}{2}\Big(1-kr^2\Big)\Big(- 2r\Big)\Big(\sin\theta\Big)^2 = -\Big(1-kr^2\Big)r\Big(\sin\theta\Big)^2$$

$$\Gamma^1_{01} = \frac{1}{2}g^{11}\Big(\partial_1 g_{10} + \partial_0 g_{11} - \partial_1 g_{01}\Big) = \frac{1}{2}g^{11}\Big(- \partial_0 g_{11}\Big)$$

$$= \frac{1}{2}\Big(- \frac{1-kr^2}{a^2}\Big)\frac{\partial}{\partial ct}\Big(- \frac{a^2}{1-kr^2}\Big)$$

$$\frac{1}{2}\Big(- \frac{1}{a^2}\Big)\frac{1}{c}\Big(- 2a\dot{a}\Big) = \frac{1}{c}\frac{\dot{a}}{a}$$

Now we know that,

$$\Gamma^2_{\mu v} = \frac{1}{2}g^{22}\Big(\partial_v g_{2\mu} + \partial_\mu g_{2v} - \partial_2 g_{\mu v}\Big) \tag{1.48}$$

$$\Gamma^2_{00} = \frac{1}{2}g^{22}\Big(\partial_0 g_{20} + \partial_0 g_{20} - \partial_2 g_{00}\Big) = \frac{1}{2}g^{22}\Big(- \partial_\theta(1)\Big)$$

$$\Gamma^2_{11} = \frac{1}{2}g^{22}\Big(\partial_1 g_{21} + \partial_1 g_{21} - \partial_2 g_{11}\Big) = \frac{1}{2}g^{22}\Big(- \partial_\theta \frac{a^2}{1-kr^2}\Big)$$

$$\Gamma^2_{22} = \frac{1}{2}g^{22}\Big(\partial_2 g_{22} + \partial_1 g_{22} - \partial_2 g_{22}\Big) = \frac{1}{2}g^{22}\Big(- \partial_\theta a^2 r^2\Big)$$

Therefore, $\Gamma^0_{00} = \Gamma^2_{11} = \Gamma^2_{22} = 0$

$$\Gamma^2_{23} = \frac{1}{2}g^{22}\Big(\partial_3 g_{23} + \partial_3 g_{23} - \partial_2 g_{33}\Big) = \frac{1}{2}g^{22}\Big(- \partial_2 g_{33}\Big)$$

$$= \frac{1}{2}\Big(- \frac{1}{a^2 r^2}\Big)\Big(- \partial_\theta\Big(- a^2 r^2\Big(\sin\theta\Big)^2\Big)$$

$$= -\Big(sin\theta\Big)\Big(cos\theta\Big)$$

$$\Gamma^2_{02} = \frac{1}{2}g^{22}\Big(\partial_2 g_{20} + \partial_0 g_{22} - \partial_2 g_{02}\Big) = \frac{1}{2}g^{22}\Big(- \partial_0 g_{22}\Big)$$

$$= \frac{1}{2}\Big(- \frac{1}{a^2 r^2}\Big)\frac{\partial}{\partial ct}\Big(- a^2 r^2\Big) = \frac{1}{c}\frac{\dot{a}}{a}$$

We know that $\Gamma^2_{02} = \Gamma^2_{20} = \frac{1}{c}\frac{\dot{a}}{a}$

$$\Gamma^2_{12} = \frac{1}{2}g^{22}\Big(\partial_2 g_{21} + \partial_1 g_{22} - \partial_2 g_{12}\Big) = \frac{1}{2}g^{22}\Big(- \partial_1 g_{22}\Big)$$

$$= \frac{1}{2}\Big(- \frac{1}{a^2 r^2}\Big)\partial_r\Big(- a^2 r^2\Big) = \frac{1}{2}\Big(\frac{1}{r^2}\Big)\Big(2r\Big) = \frac{1}{r}$$

We know that $\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}$

$$\Gamma^2_{32} = \frac{1}{2}g^{22}\Big(\partial_2 g_{23} + \partial_3 g_{22} - \partial_3 g_{32}\Big) = \frac{1}{2}g^{22}\Big(- \partial_3 g_{22}\Big) = \frac{1}{2}g^{22}\Big(\partial_\phi g_{22}\Big) = 0$$

$$\Gamma^3_{\mu v} = \frac{1}{2}\Big(\partial_v g_{3\mu} + \partial_\mu g_{3v} - \partial_3 g_{\mu v}\Big) \tag{1.49}$$

$$\Gamma^3_{\mu v} = \frac{1}{2}g^{33}\Big(\partial_\mu g_{3\mu} + \partial_\mu g_{3\mu} - \partial_3 g_{\mu v}\Big) = \frac{1}{2}g^{33}\Big(\partial_\phi g_{\mu\mu}\Big) = 0 \tag{1.50}$$

$$\Gamma^3_{03} = \frac{1}{2}g^{33}\Big(\partial_3 g_{30} + \partial_0 g_{33} - \partial_3 g_{03}\Big) = \frac{1}{2}g^{33}\Big(- \partial_0 g_{33}\Big)$$

$$= \frac{1}{2}\Big(- \frac{1}{a^2 r^2 (sin\theta)^2}\Big)\frac{\partial}{\partial ct}\Big(- a^2 r^2\Big(sin\theta\Big)^2\Big) = \frac{1}{c}\frac{\dot{a}}{a}$$

$$\Gamma^3_{13} = \frac{1}{2}g^{33}\Big(\partial_3 g_{31} + \partial_1 g_{33} - \partial_3 g_{33}\Big) = \frac{1}{2}g^{33}\Big(-\partial_1 g_{33}\Big)$$

$$= \frac{1}{2}\Big(-\frac{1}{a^2r^2(sin\theta)^2}\Big)\partial_r\Big(-a^2r^2\Big(sin\theta\Big)^2\Big) = \frac{1}{r}$$

Therefore, $\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}$

$$\Gamma^3_{23} = \frac{1}{2}g^{33}\Big(\partial_3 g_{32} + \partial_2 g_{33} - \partial_3 g_{23}\Big) = \frac{1}{2}g^{33}\Big(\partial_2 g_{33}\Big)$$

$$= \frac{1}{2}\Big(-\frac{1}{a^2r^2\Big(sin\theta\Big)^2}\Big)\partial_\theta\Big(-a^2r^2\Big(sin\theta\Big)^2\Big) = cot\theta$$

Therefore, $\Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{3} = cot\theta$

Now for Ricci Tensors:

$$R^\rho_{\sigma\mu v} = \partial_\mu\Gamma^\rho_{v\sigma} - \partial_v\Gamma^\rho_{\mu\sigma} + \Gamma^\alpha_{v\sigma}\Gamma^\rho_{\mu\sigma} - \Gamma^\beta_{v\beta} \tag{1.51}$$

$$R_{00} = R^\mu_{0\mu0} = \partial_\mu\Gamma^\mu_{00} - \partial_0\Gamma^\mu_{\mu0} + \Gamma^\alpha_{00}\Gamma^\mu_{\mu\alpha} - \Gamma^\beta_{\mu0}\Gamma^\mu_{0\beta} \tag{1.52}$$

$$= \partial_\mu(0) - \partial_0\Gamma^\mu_{\mu0} + (0)\Gamma^\mu_{\mu\alpha} - \Gamma^\beta_{\mu0}\Gamma^\mu_{0\beta}$$

$$= -\partial_0\Gamma^\mu_{\mu0} - \Gamma^\beta_{\mu0}\Gamma^\mu_{0\beta}$$

$$= -\partial_{00}\Gamma^1_{10} - \partial_0\Gamma^2_{20} - \partial_0\Gamma_{30} - \Gamma^1_{10}\Gamma^1_{01} - \Gamma^2_{20}\Gamma^2_{02} - \Gamma^3_{30}\Gamma^3_{03}$$

$$= 3\Big(-\partial_0\Gamma^1_{10} - \Gamma^1_{10}\Gamma^1_{01}\Big)$$

$$= 3\Big[-\frac{\partial}{\partial ct}\Big(\frac{1}{c}\frac{\dot{a}}{a}\Big) - \Big(\frac{1}{c}\frac{\dot{a}}{a}\Big)\Big(\frac{1}{c}\frac{\dot{a}}{a}\Big)\Big]$$

$$= -\frac{3}{c^2}\Big[\frac{\partial}{\partial t}\Big(\frac{\dot{a}}{a}\Big) + \Big(\frac{\dot{a}}{a}\Big)\Big(\frac{\dot{a}}{a}\Big)\Big]$$

$$= -\frac{3}{c^2}\Big[\frac{\partial}{\partial t}\Big(\dot{a}a^-1\Big) + \Big(\frac{\dot{a}}{a}\Big)\Big(\frac{\dot{a}}{a}\Big)\Big]$$

$$= -\frac{3}{c^2}\Big(\frac{\ddot{a}}{a} + \dot{a}(-1)(a)^{-2}\dot{a} + \Big(\frac{\dot{a}}{a}\Big)^2\Big) = -\frac{3}{c^2}\frac{\ddot{a}}{a}$$

Therefore, R_{11}, R_{22}, R_{33} are similar to each other. Hence we will get basic formulas involving $\Gamma^{\sigma}_{\mu v}$ and the compare them:

$$R_{11} = R^{\mu}_{1\mu 1} = \partial_{\mu}\Gamma^{\mu}_{11} - \partial_{1}\Gamma^{\mu}_{\mu 1} + \Gamma^{\alpha}_{11}\Gamma^{\mu}_{\mu\alpha} - \Gamma^{\beta}_{\mu 1}\Gamma^{\mu}_{1\beta} \tag{1.53}$$

$$= \partial_0\Gamma^1_{11} - \partial_1\Gamma^1_{11} - \partial_1\Gamma_{21} - \Gamma^3_{31}\Gamma^0_{\mu 0} - \Gamma^1_{10}\Gamma^2_{\mu 0} - \Gamma^3_{\mu 1}\Gamma^3_{\mu 3}$$

$$= \partial_0\Gamma^0_{11} - 2\partial_1\Gamma^2_{21} + \Gamma^0_{11}\Gamma^1_{10} + 2\Gamma^1_{11}\Gamma^2_{21} - 2\Gamma^2_{21}\Gamma^2_{22} = \frac{1}{r^2}$$

$$R_{22} = R^{\mu}_{2\mu 2} = \partial_{\mu}\Gamma^{\mu}_{22} - \partial_{2}\Gamma^{\mu}_{\mu 2} + \Gamma^{\alpha}_{22}\Gamma^{\mu}_{\mu\alpha} - \Gamma^{\beta}_{\mu 2}\Gamma^{\mu}_{2\beta} \tag{1.54}$$

$$= \partial_0\Gamma^0_{22} + \partial_1\Gamma^1_{22} - \partial_2\Gamma^3_{32} + \Gamma^0_{22}\Gamma^1_{10} + \Gamma^1_{22}\Gamma^1_{11} - \Gamma^3_{32}\Gamma^2_{23}$$

$$\partial_0\Gamma^0_{22} + \partial_1\Gamma^1_{22} + \Gamma^0_{22}\Gamma^1_{10} + \Gamma^1_{22}\Gamma^1_{11} + 1 = -1 + cot\theta$$

$$R_{33} = R^{\mu}_{3\mu 3} = \partial_{\mu}\Gamma^{\mu}_{33} - \partial_{3}\Gamma^{\mu}_{\mu 3} + \Gamma^{\alpha}_{33}\Gamma^{\mu}_{\mu\alpha} - \Gamma^{\beta}_{\mu 3}\Gamma^{\mu}_{3\beta} \tag{1.55}$$

$$= \partial_0\Gamma^0_{33} + \partial_1\Gamma^1_{33} - \partial_2\Gamma^2_{33} + \Gamma^0_{33}\Gamma^1_{10} + \Gamma^1_{33}\Gamma^2_{33} - \Gamma^2_{32}\Gamma^3_{23}$$

$$\partial_0\Gamma^0_{33} + \partial_1\Gamma^1_{33} + \Gamma^0_{33}\Gamma^1_{10} + \Gamma^1_{33}\Gamma^2_{32} = \Big(sin\theta\Big)^2$$

Now computing $\Gamma^1_{22} = \Big(sin\theta\Big)^2$ we get:

$$\partial_1\Gamma^1_{22} + \Gamma^1_{22}\Gamma^1_{11} + 1 \tag{1.56}$$

$$= \partial\Big(-\Big(1-kr^2\Big)r\Big) + \Big(-\Big(1-kr^2\Big)r\Big(\frac{kr}{1-kr^2}\Big) + 1$$

$$= 2kr^2 - 1 + kr^2 - kr^2 + 1 = 2kr^2 = 2k\gamma_{22}$$

$$R_{ii} = \frac{1}{c^2}\Big(a\ddot{a} + 2\dot{a}^2 + 2kc^2\Big)\gamma_{ii} \tag{1.57}$$

$$R_{11} = \gamma_{11}\frac{1}{c^2}\Big(a\ddot{a} + 2\dot{a}^2 + 2kc^2\Big)\Big(\frac{1}{1-kr^2}\Big) \tag{1.58}$$

$$R_{22} = \gamma_{22} = \frac{1}{c^2}\Big(a\ddot{a} + 2\dot{a}^2 + 2kc^2\Big)r^2 \tag{1.59}$$

$$R_{33} = \gamma_{33} = \frac{1}{c^2}\Big(a\ddot{a} + 2\dot{a}^2 + 2kc^2\Big)r^2\Big(sin\theta\Big)^2 \tag{1.60}$$

$$R = R^{\mu}_{\mu} = R_{\mu v}g^{\mu v} \tag{1.61}$$

$$R = R_{\mu v}g^{\mu 0} = R_{00}g^{00} + R_{11}g^{11} + R_{22}g^{22} + R_{33}g^{33}$$

$$= \frac{-1}{c^2}\frac{3a\ddot{a}}{a^2} - \frac{1}{c^2}\frac{3\Big(a\ddot{a} + 2\dot{a}^2 + 2kc^2\Big)}{a^2}$$

$$R = -6\frac{1}{c^2}\frac{a\ddot{a} + \dot{a}^2 + kc^2}{a^2} \tag{1.62}$$

Here we are assuming mostly negative for the metric convention.Therefore, $\frac{1}{2}Rg\mu v$ is same.

$$R_{\mu v} - \frac{1}{2}Rg_{\mu v} - \Lambda g_{\mu v} = \frac{8\pi G}{c^4}T_{\mu v} \tag{1.63}$$

1.4.1 Geodesics:

A path that free particles takes through spacetime with no forces acting on them. Only affected by gravity for the curvature of spacetime.

Geodesic Equation:

$$\frac{d^2\sigma}{d\lambda^2} + \Gamma^{\sigma}_{\mu v}\frac{dx^{\mu}}{d\lambda}\frac{dx^{v}}{d\lambda} = 0 \tag{1.64}$$

$$\Rightarrow \qquad 0 + \Gamma^{\sigma}_{tt}\frac{dct}{d\lambda}\frac{dct}{d\lambda} = \Gamma^{\sigma}_{tt} = \Gamma^{\sigma}_{00} = 0$$

1.4.2 FLRW Metric:

- Friedmann 1st Equation:

We know that,

$$g_{\mu v} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{(a(t))^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & -(a(t)r)^2 & 0 \\ 0 & 0 & 0 & -(a(t)r\sin\theta)^2 \end{bmatrix} \tag{1.65}$$

$$T_{\mu v} = \begin{bmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{bmatrix} \tag{1.66}$$

Now,

$$R_{00} - \frac{1}{2}Rg_{00} - \Lambda g_{00} - \frac{8\pi G}{c^4}T_{00}$$

$$\Rightarrow \qquad \frac{3}{c^2}\frac{\ddot{a}}{a}\frac{a}{a} + \frac{3}{c^2}\frac{1}{a^2}\Big(a\ddot{a} + \dot{a} + kc^2\Big) - \lambda = \frac{8\pi G}{c^2}$$

$$\Rightarrow \qquad \frac{3}{c^2}\frac{1}{a^2}\Big(\dot{a} + kc^2\Big) - \frac{\Lambda c^2}{c^2} = \frac{8\pi G}{c^2}$$

$$\Rightarrow \qquad \frac{3}{a^2}\Big(\dot{a} + kc^2\Big) = 8\pi G\rho + \Lambda c^2$$

$$\Rightarrow \qquad \frac{3\frac{1}{a^2}\Big(\dot{a}^2 + kc^2\Big)}{3} = \frac{8\pi G\rho + \Lambda c^2}{3}$$

Hence Friedmann 1st Equation:

$$\frac{\dot{a}^2 + kc^2}{a^2} = \frac{8\pi G\rho + \Lambda c^2}{3} \tag{1.67}$$

- Friedmann 2nd Equation :

Now,

$$R_{\mu v}g^{\mu v} - \frac{1}{2}Rg_{\mu v}g^{\mu v} - \Lambda g_{\mu v}g^{\mu v} = \frac{8\pi G}{c^4}T_{\mu v}T^{\mu}_{\mu} \tag{1.68}$$

$$\Rightarrow \qquad R^{\mu}_{\mu} - \frac{1}{2}R\delta^{\mu}_{\mu} - \Lambda\delta^{\mu}_{\mu} = \frac{8\pi G}{c^4}T^{\mu}_{\mu}$$

$$\Rightarrow \qquad R - 2R - \Lambda 4 = \frac{8\pi G}{c^4}T$$

$$\Rightarrow \qquad -R - 4\Lambda = \frac{8\pi G}{c^4}T$$

$$\Rightarrow \qquad \Big(-\Big(-6\frac{1}{c^2a^2}\Big(a\ddot{a} + \dot{a}^2 + kc^2\Big)\Big) - 4\Lambda = \frac{8\pi G}{c^4}\Big(\rho c^2 - 3p\Big)$$

$$\Rightarrow \qquad \frac{6}{c^2}\Big(\frac{a\ddot{a}}{a^2}+\frac{\dot{a}^2+kc^2}{a^2}\Big)-4\Lambda=\frac{8\pi G}{c^4}\Big(\rho c^2-3p\Big)$$

$$\Rightarrow \qquad \Big(\frac{a\ddot{a}}{a^2}+\frac{\ddot{a}^2+kc^2}{a^2}\Big)-\frac{4}{6}\Lambda c^2==\frac{8\pi G}{c^4}\Big(\rho c^2-3p\Big)$$

$$\Rightarrow \qquad \frac{\ddot{a}}{a}+\frac{\dot{a}^2+kc^2}{a^2}-\frac{2}{3}\Lambda c^2=\frac{4\pi G}{3}\Big(\rho c^2-3p\Big)$$

$$\Rightarrow \qquad \frac{\ddot{a}}{a}+\frac{8\pi G\rho}{3}+\frac{\Lambda c^2}{3}-\frac{2}{3}\Lambda c^2=\frac{4\pi G}{3}\Big(\rho c^2-3p\Big)$$

$$\Rightarrow \qquad \frac{\ddot{a}}{a}=-\frac{1}{3}\Lambda c^2+\frac{2}{3}\Lambda c^2-\frac{8\pi G\rho}{3}+\frac{4\pi G}{3}\rho$$

$$\frac{\ddot{a}}{a}=\frac{1}{3}\Lambda c^2-\frac{4\pi G}{3}\Big(\rho+\frac{p}{c^2}\Big) \tag{1.69}$$

In the Friedmann equations p, ρ, k, Λ all the values the results is consistent with General Relativity.

For the Static universe $\dot{a}=\ddot{a}=0$

$$\frac{\ddot{a}}{a}=-\frac{4\pi G}{3}\Big(\rho+\frac{3p}{c^2}\Big)+\frac{\Lambda c^2}{3}$$

$$\Rightarrow \qquad \frac{4\pi G}{3}\Big(\rho+\frac{3p}{c^2}\Big)+\frac{\Lambda c^2}{3}=\Lambda c^2$$

$$\Lambda=\frac{4\pi G}{c^2}\Big(\rho+\frac{3p}{c^2}\Big) \tag{1.70}$$

Now,

$$\frac{\dot{a}^2+kc^2}{a^2}=\frac{8\pi G\rho+\Lambda c^2}{3}$$

$$\Rightarrow \qquad \frac{kc^2}{a^2}=\frac{8\pi G\rho+4\pi G\Big(\rho+\frac{3p}{c^2}\Big)}{3}$$

$$\Rightarrow \qquad \frac{kc^2}{a^2}=\frac{12\pi G\rho+12\pi G\Big(\frac{p}{c^2}\Big)}{3}$$

$$\Rightarrow \qquad \frac{k}{a^2}=\frac{4\pi G}{c^2}\Big(\rho+\frac{p}{c^2}\Big) \tag{1.71}$$

Einstein static universe: $\dot{a} = \ddot{a} = p = 0$

$$\Lambda = \frac{4\pi G}{c^2}\rho$$

$$\frac{k}{a^2} = \frac{4\pi G}{c^2}\rho$$

Here Λ is a cosmological constant .

(1) If Matter $\rho(a) \propto a^{-3}$ then the energy is conserved as universe is expanding.

(2) If Radiation $\rho(a) \propto a^{-4}$ then the energy is lost as universe is expanding.

(3) If Dark Energy $\rho(a) \propto 1$ the the energy is gained as universe is expanding.

Therefore, Energy is conserved in General Theory of Relativity.

1.5 Dark Energy Models:

As we have studied earlier the cosmological constant Λ was originally introduced by Einstein in 1917 to achieve a static universe. As we know that the Hubble discovered the expansion of our universe in 1929 which was stated by Einstein that was no longer required. Therefore, the cosmological constant naturally arises from the energy density of the vacuum. However the Λ is much greater than the Hubble constant H_0, which was originate from the vacuum energy. Thus it is known as cosmological constant which was exist before the expansion of the accelerated universe. There is having lots of way to solve the problems.

Fine tuning problem was originated from the energy density, which was suffers from fine-tuning method having Λ and H_0. We know that the contribution related to the ordering of fields may disappears when it is adopted. The procedure to required for fine-tuning and the energy density of the universe to observe. The resolution at the zero point of the energy is super-symmetry. The sense of theories do not admit a non-zero cosmological constant. Hence the presence of scalar field have the standard model in the particle physics has the probability to ensure that the spontaneous breakdown

of the gauge symmetry. This scalar field is known as hierarchy problem and it lies between the quadratic nature of the divergence of the scalar fields mentioned dreams of a harmonic partner for every boson and vice versa. Since the loops comes with all the negative sign. The super-partners of the particles degenerate and wants to ensure that no new scalars of the standard model particles in Tev. Different ruled out of different results have the specific bound of the masses from the analysis of different experimental data. Hence the cosmological problems leads to many authors and have different approach of each and every dark energy issues.

The nature of both dark energy and dark matter is unknown and the opinion assumes on the dark energy of the dark matter modeled is non-relativistic fluid. The fact of energy densities of dark energy and dark matter have the inverse of third power of the cosmic scale factor which requires have extremely fine-tuned and the initial condition in the early universe. In every early universe and in the far future universe of the energy densities and have differ many orders of magnitude. This is known as Cosmological Coincidence Problem. It ensures the ratio of dark energy and dark matter, and the present periods appears in the singled out by a ratio of the order of one and motivated the creation of different dark energy models. The simple constraints of the static universe is "constant" to prevent the formation of galaxies. The phenomenological models which was demonstrated as vacuum are no longer constant and changes with time in non-gravitational interactions with dark matter. The universe approach the stationary periodic solution and have the finite value of density parameters which was observed in the CCP. This represents different kinds of graphic in a simple way. The expansion of the universe is homogeneous and isotropic and the cosmic time of this parametrization is less dramatic with only one respect. The parameterized in terms of redshift z is have a special period in cosmic history.
There are different Dark-Energy Models[36]:

- Quintessence[53]

- K-essence [54]
- Tachyon field [52]
- Phantom (ghost) field [53]
- Chaplygin gas [54]
- Holographic Dark Energy Model [02]
- Agegraphic Dark REnergy Model [55]

  etc...

2

Modified Gravity Models

2.1 Prelude

2.2 f(Q) gravity:

This section aims to discuss the basic equation in $f(Q)$ gravity.

Here we will consider and elaborate the equation for f(Q) gravity as the related equation.Now we will discuss the f(Q) gravity as [55]:-

$$S = \int \left[\frac{-1}{2\kappa^2} f(Q) + \mathcal{L}_{\Updownarrow}\right] \sqrt{-g} d^4 \tag{2.1}$$

here L_m consider as Lagrangian,$f(Q)$ is known as arbitrary function for the non-metricity scalar Q and g is the determinant for the metric tensor $g_{\mu\nu}$ and the non-metricity scalar be defined as:-

$$Q = \frac{-1}{4} Q_{\alpha\beta\gamma} Q^{\alpha\beta\gamma} + \frac{1}{2} Q_{\alpha\beta\gamma} Q^{\gamma} \beta\alpha + \frac{1}{4} Q_{\alpha} Q^{\alpha} - \frac{1}{2} Q_{\alpha} \tilde{Q}^{\alpha} \tag{2.2}$$

where,

$$Q_{\alpha} \equiv Q^{\mu}_{\alpha\mu} \tag{2.3}$$

$$Q \equiv Q^{\mu\alpha_\mu} \tag{2.4}$$

$$Q \equiv \nabla_\alpha g_{\mu\nu} \tag{2.5}$$

Now we will take $f(Q) = Q$ then we determined the Symmetric Teleparallel Equivalent of General Relativity.We get the equations from equations (2.1),(2.2),(2.3),(2.4),(2.5) which will be given as

$$\frac{2}{\sqrt{-g}}\nabla_\alpha\sqrt{-g}g_{\beta\nu}f_Q\Big[-\frac{1}{2}L^{\alpha\mu\beta}+\frac{1}{4}g^{\mu\beta}\Big(Q^\alpha-\tilde{Q}^\alpha\Big)-\frac{1}{8}\Big(g^{\alpha\mu}Q^\beta+g^{\alpha\mu}+g^{\alpha\beta}Q^\nu\Big)\Big]+f_Q$$

$$\Big[-\frac{1}{2}L^{\mu\alpha\beta}\frac{-1}{8}\Big(g^{\mu\alpha}Q^\beta+g^{\mu\beta}Q^\alpha\Big)+\frac{1}{4}g^{\alpha\beta}\Big(Q^\mu-\tilde{Q}^\mu\Big)\Big]Q_{\nu\alpha\beta}+\frac{1}{2}\delta^{\mu_\nu}f=\kappa^2T^\mu_\nu \tag{2.6}$$

The deformation of tensor which we will get from:-

$$L^{\alpha_{\mu\nu}}=\frac{1}{2}Q^\alpha_{\mu\nu}-Q^\alpha_{\mu\nu} \tag{2.7}$$

$f(Q)$ gravity examined the phenomenological analysis of the cosmic observables within the framework of $f(Q)$ gravity in light of the degeneracy between the modified gravity parameter and neutrino physics, and we pinpoint particular patterns that enable the breaking of this degeneracy [66].

2.3 f(R) Gravity :

From the Einstein general theory of relativity the Lagrangian for geometry is $L_G = R$ which can be generalized. The $f(R)$ theory of gravity is the modification of General Relativity which is the action of Einstein-Hilbert and is generalized by substituting the Ricci scalar R with a function $f(R)$. This manner creates additional freedom that can account for cosmic acceleration without dark energy [26].

$$f(R) = ... + \frac{\alpha_2}{R} + \frac{\alpha_1}{R} - 2\Lambda + R + \frac{R^2}{\beta_2} + \frac{R^3}{\beta_3} + ...$$

here, α_i and β_i are the coefficients.

The Mathematical Formulation:

The action for $f(R)$ gravity is:

$$I = \int \sqrt{-g}\Big[f(R) + 2L_m\Big]d^4x$$

where L_m is known as Lagrangian. And vary with respect to $g_{\mu v}$ and we will get Einstein Field Equation:

$$F(R)R_{\mu v} - \frac{1}{2}g_{\mu v}f(R) - \Big[\nabla_\mu\nabla_v - g_{\mu v}\Box\Big]F(R) = -kT_{\mu v}$$

where, $F(R) = \frac{df(R)}{dR}$, ∇_μ is covariant derivative. Hence,

$$\Box F = \nabla^\mu\nabla_\mu F = \frac{1}{\sqrt{-g}}\partial_\mu\Big(\sqrt{-g}g^{\mu v}\partial_v F\Big)$$

Here $T_{\mu v}$ is the energy momentum tensor. The trace field equation is:

$$3\Box F(R) + F(R)R - 2f(R) = -kT$$

Here, $T = g^{\mu v}T_{\mu v}$.

This resorts to an altered field equation, one that augments the gravitational dynamics and attempt to explain the acceleration of the universe without involving the dark energy. Without cosmological constant we will take $f(R) = R$ which gives $F(R) = 1$, $\Box F = 0$ and $R = kT$. Hence the field equation in $f(R)$ will give different solution from the Einstein theory of field equation. Therefore $T = 0$ and $R = 0$ will be taken as constant. $f(R)$ gravity models have been examined the improve the general relativity and examine the current state of the art in $f(R)$ theories of gravity which have been put forth as a substitute for dark energy in the explanation of cosmic acceleration. The difficulties posed by limited conditions for its feasibility are examined along with the achievements of $f(R)$ gravity [51]

2.4 Gauss-Bonnet Gravity :

We consider another form of gravity which is the modification of Einstein-Hilbert action known as Gauss Bonnet Gravity (GB) and termed as [26]:

$$G = R^2 - 4R^{\mu v}R_{\mu v} + R^{muv\rho\sigma}R_{\mu v\rho\sigma}$$

which was recommended by Carl Friedrich Gauss and Pierre Ossian Bonnet. The general theory of action is:

$$I = \int \sqrt{-g}\Big[R - 2\Lambda + \alpha_{GB}L_{GB} + 2kL_m\Big]d^4x$$

Here, $L_{GB} = R^2 - 4R^{\mu v}R_{\mu v} + R^{muv\rho\sigma}R_{\mu v\rho\sigma}$ where α_{GB} is considered as a coupling constant which is commonly known as GB parmeter. We assume $\alpha_{GB} \to 0$ and again we will come around to Einstein gravity. This modified gravity is an arbitrary function which is commonly termed as topological invariant and added to the action of General relativity. The field equation of GB gravity is:

$$G_{\mu v} + H_{\mu v} = -kT_{\mu v}$$

Here H is known as Einstein Hilbert action and T is known as energy-momentum tensor.

Hence,

$$G_{\mu v} = R_{\mu v} - \frac{1}{2}g_{muv}R + \Lambda g_{\mu v}$$

$$H_{\mu v} = 2\Big(R_{\mu\lambda\rho\sigma}R_{v}^{\lambda\rho\sigma} - R_{\mu\rho v\sigma}R^{\rho\sigma} - 2R_{\mu\sigma}R_{v}^{\sigma}\Big) - \frac{1}{2}g_{\mu v}L_{GB}$$

Gauss-Bonnet gravity models investigated the spontaneous scalarization of black-hole. Coupling of matter to gravitation we studied by GBM Models.This has shown positive results showing the presence of gravity [61]

2.5 f(T) Gravity:

From the Einstein general theory of relativity we consider the another form of modified gravity is known as $f(T)$ gravity. It elaborate the accelerating and expansion of our universe. Here we will considered a type of model commonly known as Lagrangian density which is expanded as a function of $f(T)$. It is not having any Lorentz symmetry. The general action of f(T) gravity is [42]:

$$dS^2 = g_{\mu v}dx^{\mu}dx^{v}$$

$$dx^{\mu} = e_i^{\mu}\theta^i, \qquad \theta^i = e_{\mu}^i dx^{\mu}$$

The torsion tensor which is antisymmetric can be written in the form of

$$T_{\mu v}^{\alpha} = \Gamma_{v\mu}^{\alpha} - \Gamma_{\mu v}^{\alpha}$$

The torsion scalar:

$$T = T_{\mu v}^{\alpha} S_{\alpha}^{\mu v}$$

Hence the field equation is :

$$S_{\mu}^{v\rho}\partial_{\rho}T f_{TT} + \left[e^{-1}e_{\mu}^{i}\partial_{\rho}\Big(ee_i^{\alpha}S_{\alpha}^{v\rho}\Big) + T_{\lambda\mu}^{\alpha}S_{\alpha}^{v\lambda}\right]f_T + \frac{1}{4}\delta_{\mu}^{v}f = -\frac{k}{2}T_{\mu}^{v}$$

This gravity is the generalization of teleparallel gravity which arise in the form of torsion of spacetime instead in the form of curvature. $f(T)$ gravity models have been studied extensively in literature the influence of torsion in gravity has been thoroughly explored with the primary aim of aligning gravity more closely with its gauge formulation and integrating spin into a geometric framework [59].

3

A Cosmological Holographic Reconstruction of $f(Q)$ Gravity Models

3.1 Prelude

In this chapter the published research paper has been thoroughly reviewed entitled below.

"A Cosmological Holographic Reconstruction of f(Q) Theory"

Here we will discuss about two dark energy models namely: (i)Grand-Oliverous model (ii)Chen-Jing model that lead to different $f(Q)$ gravity models. We will discuss different cases and study how the universe behave after the modification of the gravity.

3.2 Basic Assumptions:

We will assume the homogeneous and isotropic Friedmann-Lemaitre-Walker(FLRW) metric for our further calculation

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \tag{3.1}$$

Here,a(t) will be considered as scalar factor in reference with the cosmic time t.Therefore the field eqations 2.6 for $f(Q)$ gravity (i.e,$Q = 6H^2$) for the framework in FLRW space-

time decrease to

$$3H^2 = \kappa\Big(\rho_m + \rho_{de}\Big) \tag{3.2}$$

$$3H^2 + \dot{H} = -\kappa\Big(p_m + p_{de}\Big) \tag{3.3}$$

Here H is considered as the Hubble parameter which is demonstrated as $H = \frac{\dot{a}}{a}$.Therefore the dark energy and pressure both will be considered as FLRW equation which was identified and can be written as

$$\rho_{de} = \frac{3}{\kappa^2}H^2\Big(1 - 2f_Q\Big) + \frac{f}{6}\Big] = \frac{3}{\kappa^2}\Big[\frac{Q}{6}\Big(1 - 2f_Q\Big) + \frac{f}{6}\Big] \tag{3.4}$$

$$p_{de} = \frac{-1}{\kappa^2}\Big[2\dot{H}\Big(1 - 2f_Q\Big) + \frac{f}{2} + 3H^2\Big(1 - 8f_{QQ}\dot{H} - 2f_Q\Big)\Big] \tag{3.5}$$

Where ρ_m will be considered as density energy of matter and p_m will be our pressure from the matter fluid,and a dot is being represented here as derivative with respect to its cosmic for the time t.The the equation conservation for the matter for any components will be considered as

$$\dot{\rho}_m + 3H\Big(\rho_m + p_m\Big) = 0 \tag{3.6}$$

3.3 Considered Dark Energy Models;

The cosmological

$$24f_{QQ}\dot{H}H^2 - f < 6\dot{H}\Big(1 - f_Q\Big) \tag{3.7}$$

The relation help us to identified the dimensions of $f(Q)$ models which will be behave as dark energy.The the Friedmann equation determinate as

$$8\rho = \Sigma_i A\Big(X, Y, ...\Big)\frac{\partial^n}{f}\Big(X, Y, ...^{n_1^{n_2}}\Big) \tag{3.8}$$

Granda Oliveros cut-off:

$$L_{GO} = \Big(\alpha H^2 + \beta\dot{H}\Big)^{\frac{-1}{2}} \tag{3.9}$$

Here the Hubble parameter is denoted as $\frac{\dot{a}}{a}$ where 'dot' gives is the derivatives of the cosmic with respect to time t.The model was denoted by the two parameter α and

β.The extension of the universe impute the presence of dark energy,and the relativity for the rate of change of Hubble parameter(H) has been considered as obvious.Hence we will consider the function as H and $\dot{H}$.Then the relativity has been controlled by the parameters α and β.The holographic dark energy at the particle from the horizon which can be defined as expeditious way to enlarge our present universe.Henceforth,we will take IR cut off and the horizon from the future and described our problems which can be determined from our universe.From this work,we will work on the modification of Granda-Oliveros model which will demonstrate for dark energy.We know that L_{GO} and the value which was limited as case α, β=2,1.Hence the non flat universe from the observation data is,[44]

$$\alpha = 0.8824^{+0.2180}_{-0.1163}(1\sigma)^{+0.1247}_{-0.1102}(2\sigma) \qquad \beta = 0.5016^{+0.0973}_{-0.0871}(1\sigma)^{+0.1247}_{-0.1102}(2\sigma) \tag{3.10}$$

And from the flat universe the observation is,[44]

$$\alpha = 0.8502^{+0.0984}_{-0.0875}(1\sigma)^{+0.1299}_{-0.1064}(2\sigma) \qquad \beta = 0.4817^{+0.0842}_{-0.0773}(1\sigma)^{+0.1176}_{-0.0955}(2\sigma) \tag{3.11}$$

- **Granda-Oliveros Dark Energy Model:**

  Granda-Oliveros model for the energy density is

  $$\rho_{GODE} = 3c^2\Big(\alpha H^2 + \beta\dot{H}\Big) \tag{3.12}$$

  Here the α and β is taken as dimensionless parameter.

- **Chen-Jing Dark Energy model** Chen-Jing model was introduced in literature as a generalization of Granda-Oliveros Dark Energy model.The energy density for this model reads[22, 23]:

  $$\rho_{CJDE} = 3c^2\Big[\alpha\Big(\frac{\ddot{H}}{H}\Big) + \beta\dot{H} + \gamma H^2\Big] \tag{3.13}$$

here α, β, γ is known as dimensionless parameters.If α=0 then the model will consider as if α=1, β=1, $\gamma = 2$ will be reciprocal for the proportion to IR where c will be considered as dimensionless parameter for unit order.

3.4 The reconstruction Schemes:

3.4.1 Case I

Power-law form of scale factor[51]:

$$a(t) = a_0 t^n \tag{3.14}$$

Here we will assume $m > 0$ will be a constant and a_0 which is the function of a_t[50, 18].This solution defines and gives the solution in the horzon section and the universe which is expanding gives the flatness solution to inspect different types of modification in the $f(Q)$ gravity and we will solve using different ordinary and partial differential equation respectively. The Hubble parameter is given by $H(t) = \frac{\dot{a}}{a}$. Using equation(1.20) in equation(1.29) we get

$$H = \frac{m}{t}, \qquad \dot{H} = -\frac{m}{t^2} \tag{3.15}$$

$Q = 6H^2$

$$Q = 6\frac{m^2}{t^2}$$

$$t^2 = 6\frac{m^2}{Q}$$

$$t = m\sqrt{\frac{6}{Q}} \tag{3.16}$$

Therefore,

$$H = \sqrt{\frac{Q}{6}}$$

$$\dot{H} = \frac{-Q}{6m}$$

$$\dot{H} = \frac{-Q}{6m}\text{ ,} \qquad \ddot{H} = \frac{Q}{3m^2}\sqrt{\frac{Q}{6}} \tag{3.17}$$

From the given Friedmann equation for the given $f(Q)$ gravity will be taken as:-

$$3H^2 = \kappa^2\Big(\rho_m + \rho_{de}\Big) \tag{3.18}$$

- **Grand-Oliveros model:** The energy density of Granda-Oliveros dark energy for power-law form of scale factor takes the form:-

$$3c^2\Big(\frac{\alpha}{6}-\frac{\beta}{6m}\Big)Q \tag{3.19}$$

Comparing the energy density of $f(Q)$ gravity and Granda-Oliveros dark energy model from equation(3.14) and(3.15) we have

$$\frac{3}{\kappa^2}\Big[\frac{Q}{6}\Big(1-2f_Q\Big)+\frac{f}{6}\Big]=3c^2\Big(\frac{\alpha}{6}-\frac{\beta}{6m}\Big)Q$$

$$\Rightarrow \quad Q\Big(1-2f_Q\Big)+f=\kappa^2c^2\Big(\alpha-\frac{\beta}{m}\Big)Q$$

$$\Rightarrow \quad Q-2Qf_Q+f=\frac{\kappa^2c^2}{m}\Big(\alpha m-\beta\Big)Q$$

$$\Rightarrow \quad 2Qf_Q-f=\Big[1-\frac{\kappa^2c^2}{m}\Big(\alpha m-\beta\Big)\Big]Q$$

$$\Rightarrow \quad 2Qf'(Q)-f(Q)=\Big[1-\frac{\kappa^2c^2}{m}\Big(\alpha m-\beta\Big)\Big]Q$$

$$\Rightarrow \quad f'(Q)-\frac{1}{2Q}f(Q)=\Big[1-\frac{\kappa^2c^2}{m}\Big(\alpha m-\beta\Big)\Big]Q \tag{3.20}$$

Integrating factor of the above equation is:

$$e^{\int -\frac{1}{2Q}dQ}=\frac{1}{\sqrt{Q}} \tag{3.21}$$

Multiplying equation(3.20) by the integrating factor and integrating we have:

$$f(Q)\frac{1}{\sqrt{Q}}=\frac{1}{2}\Big[1-\frac{\kappa^2c^2}{m}\Big(\alpha m-\beta\Big)\Big]\sqrt{Q}+c_1$$

$$\Rightarrow \quad f(Q)=c_1\sqrt{Q}\Big[1-\frac{\kappa^2c^2}{m}\Big(\alpha m-\beta\Big)\Big]Q, \quad \text{where } c_1 \text{ is an integrating factor.} \tag{3.22}$$

- **Chen-Jing model:** The energy density for Chen-Jing model is given by;

$$\rho_D=3c^2\Big[\alpha\Big(\frac{\ddot{H}}{H}\Big)+\beta\dot{H}+\gamma H^2\Big]$$

$$\rho_D = 3c^2\Big[\alpha\frac{Q}{3m^2} - \frac{\beta Q}{6m} + \frac{\gamma Q}{6}\Big] \tag{3.23}$$

Now for the reconstruction we have;

$$\frac{3}{\kappa^2}\Big[\frac{Q}{6}(1-2f_Q) + \frac{f}{6}\Big] = 3c^2\Big[\alpha\frac{Q}{3m^2} - \frac{\beta Q}{6m} + \frac{\gamma Q}{6}\Big]$$

$$\Rightarrow \qquad Q\Big(1-2f_Q\Big) + f = \kappa^2c^2\Big[\frac{2d}{m^2} - \frac{\beta}{m} + \gamma\Big]Q$$

$$\Rightarrow \qquad Q - 2Qf_Q + f = \frac{\kappa^2c^2}{m^2}\Big[2\alpha - m\beta + \gamma m^2\Big]Q$$

$$\Rightarrow \qquad Qf_Q - f = \frac{1}{2}(1 - \frac{\kappa^2c^2}{m^2})\Big[2\alpha - m\beta + \gamma m^2\Big]$$

$$\Rightarrow \qquad f'(Q) - \frac{1}{2Q} = \frac{1}{2}\Big[1 - \frac{\kappa^2c^2}{m^2}\Big(2\alpha - m\beta + m^2\gamma\Big)\Big] \tag{3.24}$$

Integrating factor of the above equation is:

$$e^{\int -\frac{1}{2Q}dQ} = \frac{1}{\sqrt{Q}} \tag{3.25}$$

Multiplying equation(3.24) and integrating factor and integrating we have:

$$f(Q)\frac{1}{\sqrt{Q}} = \frac{1}{2}\Big[1 - \frac{\kappa^2c^2}{m^2}\Big(2\alpha - m\beta + m^2\gamma\Big)\Big]\int\frac{dQ}{\sqrt{Q}}$$

$$\Rightarrow \qquad f(Q)\frac{1}{\sqrt{Q}} = \Big[1 - \frac{\kappa^2c^2}{m^2}\Big(2\alpha - m\beta + m^2\gamma\Big)\Big]\sqrt{Q} + c_2$$

$$\Rightarrow \qquad f(Q) = c_2\sqrt{Q} + \Big[1 - \frac{\kappa^2c^2}{m^2}\Big(2\alpha - m\beta + m^2\gamma\Big)\Big]Q \tag{3.26}$$

3.4.2 Case II

Considering the scale factor as:

$$a(t) = a_0(t_s - t)^{-n} \tag{3.27}$$

Where we will consider as $n > 0$ as a constant and $t < t_s$ here t_s considered as the future singularity for the finite time.Therefore we will solve the equation using differential equation for f(Q) gravity :

$$H = \frac{n}{t_s - t}, \qquad \dot{H} = 0$$

$Q = 6H^2$

$$Q = \frac{6n^2}{t_s - t}^2$$

Therefore,

$$H = n\frac{\sqrt{6n^2}}{Qt} \tag{3.28}$$

$$\dot{H} = n \tag{3.29}$$

From the given Friedmann equation for the $f(Q)$ gravity is taken as:

$$3H^2 = \kappa^2(\rho_m + \rho_{de})$$

- **Grand-Oliveros model:** The energy density of Granda-Oliveros dark energy for power-law form of scale factor takes the form:-

$$\rho_D = 3c^2\Big(\alpha H^2 + \beta\dot{H}\Big)$$

$$\Rightarrow \qquad 3c^2\Big(\frac{\alpha}{6} - \frac{\beta}{6n}\Big)Q \tag{3.30}$$

Comparing the energy density of $f(Q)$ gravity and Granda-Oliveros dark energy model from the equations (3.30)we have

$$\frac{3}{\kappa^2}\Big[\frac{Q}{6}(1 - 2f_Q) + \frac{f}{6}\Big] = 3c^2\Big(\frac{\alpha}{6} - \frac{\beta}{6m}\Big)Q$$

$$\Rightarrow \qquad Q\Big(1 - 2f_Q\Big) + f = \kappa^2c^2\Big(\alpha + \frac{\beta}{n}\Big)$$

$$\Rightarrow \qquad Q - 2f_Q + f = \Big[1 - \frac{\kappa^2c^2}{n}\Big(\alpha n + \beta\Big)\Big]Q$$

$$\Rightarrow \qquad 2Qf'(Q) - f(Q) = \Big[1 - \frac{\kappa^2c^2}{n}\Big(\alpha n + \beta\Big)\Big]Q$$

$$\Rightarrow \qquad f'(Q) - \frac{1}{2Q}f(Q) = \frac{1}{2}\Big[1 - \frac{\kappa^2c^2}{n}\Big(\alpha n + \beta\Big)\Big] \tag{3.31}$$

Integrating factor of the above equation is:

$$e^{\int -\frac{1}{2Q}dQ} = \frac{1}{\sqrt{Q}}$$

Multiplying equation(3.31) by the integrating factor and integrating we have:

$$\Rightarrow \qquad f(Q)\frac{1}{\sqrt{Q}} = \frac{1}{2}\Big[1 - \frac{\kappa^2 c^2}{n}\Big(\alpha n + \beta\Big)\Big]\int \frac{1}{\sqrt{Q}}dQ$$

$$\Rightarrow \qquad f(Q)\frac{1}{\sqrt{Q}} = \Big[1 - \frac{\kappa^2 c^2}{n}\Big(\alpha n + \beta\Big)\Big]\sqrt{Q} + c_3$$

$$f(Q) = c_3\sqrt{Q} + \Big[1 - \frac{\kappa^2 c^2}{n}(\alpha n + \beta)\Big]Q \tag{3.32}$$

- **Chen-Jing model:** The energy density for Chen-Jing model is given by;

$$\rho_D = 3c^2\Big[\alpha\Big(\frac{\ddot{H}}{H}\Big) + \beta\dot{H} + \gamma H^2\Big]$$

$$\Rightarrow \qquad \rho_D = 3c^2\Big[\frac{\alpha Q}{3n^2} + \frac{\beta Q}{6n} + \frac{\gamma Q}{6}\Big] \tag{3.33}$$

Now for the reconstruction we have;

$$\frac{3}{\kappa^2}\Big[\frac{Q}{6}(1 - 2f_Q) + \frac{f}{6}\Big] = 3c^2\Big[\alpha\frac{Q}{3n^2} - \frac{\beta Q}{6n} + \frac{\gamma Q}{6}\Big]$$

$$\Rightarrow \qquad Q(1 - 2f_Q) + f = \kappa^2 c^2\Big[\frac{2\alpha}{n^2} + \frac{\beta}{n} + \gamma\Big]Q$$

$$\Rightarrow \qquad Q - 2f_Q + f = \frac{\kappa^2 c^2}{n^2}[2\alpha + n\beta + \gamma n^2]Q$$

$$\Rightarrow \qquad Q - f = \frac{1}{2}\Big[1 - \frac{\kappa^2 c^2}{n^2}\Big[2\alpha + n\beta + \gamma n^2\Big]Q$$

$$\Rightarrow \qquad f_Q - \frac{1}{2Q} = \frac{1}{2}\Big[1 - \frac{\kappa^2 c^2}{n^2}\Big(2\alpha + n\beta + \gamma n^2\Big)\Big]$$

$$\Rightarrow \qquad f'(Q) - \frac{1}{2}f(Q) = \frac{1}{2}\Big[1 - \frac{\kappa^2 c^2}{n^2}\Big(2\alpha + n\beta + n^2\gamma\Big)\Big] \tag{3.34}$$

Integrating factor of the above equation is:

$$e^{\int -\frac{1}{2Q}dQ} = \frac{1}{\sqrt{Q}}$$

Multiplying equation(3.34) the integrating factor and integrating we have:

$$\Rightarrow \qquad f(Q)\frac{1}{\sqrt{Q}} = \frac{1}{2}\Big[1 - \frac{\kappa^2 c^2}{n^2}\Big(2\alpha + n\beta + \gamma n^2\Big)\Big]\int \frac{dQ}{\sqrt{Q}}$$

$$\Rightarrow \qquad f(Q)\frac{1}{\sqrt{Q}} = \Big[1 - \frac{\kappa^2 c^2}{n^2}\Big(2\alpha + n\beta + \gamma n^2\Big)\Big]\sqrt{Q} + c_4$$

$$f(Q) = c_4\sqrt{Q} + \Big[1 - \frac{c^2\kappa^2}{n^2}\Big(2\alpha + \beta n + \gamma n^2\Big)\Big]Q \tag{3.35}$$

3.4.3 Case III

Here we will consider the exponential form as [44, 46]:

$$a(t) = e^{Bt^{\theta}} \tag{3.36}$$

Here $0 < \theta < 1$ and $B > 0$ will be considered as constants and solve the differential equation for $f(Q)$ gravity as:

$$H = B\theta t^{\theta-1}, \qquad \dot{H} = B\theta\Big(\theta-1\Big)t^{\theta-2}, \qquad \ddot{H} = \theta\Big(\theta-1\Big)\Big(\theta-2\Big)t^{\theta-3}$$

Again for $f(Q)$ gravity we will take $Q = 6H^2$ and solve the equations;

$$H = \sqrt{\frac{Q}{6}}, \qquad \dot{H} = B\theta\Big(\theta-1\Big)\Big(\frac{\sqrt{\frac{Q}{6}}}{B\theta}\Big)^{\frac{\theta-2}{\theta-1}}, \qquad \ddot{H} = \frac{1}{B}\Big(\theta-1\Big)\Big(\theta-2\Big)\theta^{\frac{2}{\theta-1}}\Big(\sqrt{\frac{Q}{6}}\Big) \tag{3.37}$$

Now from the given Friedmann equation for the given f(Q) gravity will be taken as:-

$$3H^2 = \kappa^2\Big(\rho_m + \rho_{de}\Big)$$

- **Granda-Oliverous model:**

The energy density of Granda-Oliveros dark energy for exponential form of scale factor takes the form:-

$$\rho_D = 3c^2\Big(\alpha H^2 + \beta\dot{H}\Big)$$

$$\rho_D = 3c^2\Big(\frac{\alpha}{6} + \beta\frac{B\theta(\theta-1)}{B\theta}\Big)^{\frac{\theta-2}{\theta-1}}\frac{\sqrt{Q}}{6} \tag{3.38}$$

Comparing the energy density of $f(Q)$ gravity and Granda-Oliveros dark energy model from the equation(3.38)we have

$$\frac{3}{\kappa^2}\Big[\frac{Q}{6}\Big(1-2f_Q\Big) + \frac{f}{6}\Big] = 3c^2\Big(\frac{\alpha}{6} + \beta\frac{B\theta\Big(\theta-1\Big)}{B\theta}\Big)\sqrt{\frac{Q}{6}}$$

$$\Rightarrow \qquad Q\Big(1-2f_Q\Big) + f = \Big[\kappa^2c^2\Big(\frac{\alpha}{6} - \beta\frac{B\theta\Big(\theta-1\Big)}{B\theta}\Big)\Big]\sqrt{\frac{Q}{6}}$$

$$\Rightarrow \qquad Q\Big(1-2f_Q\Big)+f=\Big[\kappa^2c^2\Big(\alpha-\beta\Big)\Big(\frac{\alpha}{6}-\beta\frac{B\theta\Big(\theta-1\Big)}{B\theta}\Big)\Big]\sqrt{\frac{Q}{6}}$$

$$\Rightarrow \qquad Q-2Qf_Q-f=\Big[\kappa^2c^2\Big(\alpha-\beta\Big)-c^2\kappa^2\beta\Big(B\theta\Big)^{\frac{1}{\theta-1}}\Big(\frac{Q}{6}\Big)^{\frac{\theta-2}{2(\theta-1)}}-c^2\kappa^2\beta B\Big]$$

$$\Rightarrow \qquad 2Qf'(Q)-f(Q)=\Big[1-\kappa^2c^2(\alpha-\beta)-c^2\kappa^2\beta\Big(B\theta)^{\frac{1}{\theta-1}}(\frac{\theta-2}{2(\theta-1)}\Big)-c^2\kappa^2\beta B\Big]\sqrt{\frac{Q}{6}}$$

$$\Rightarrow \qquad f'(Q)-\frac{1}{2Q}f(Q)=\frac{1}{2}\Big[1-\kappa^2c^2\Big(\alpha-\beta\Big)-c^2\kappa^2\beta\Big((B\theta)^{\frac{1}{\theta-1}}\Big(\frac{\theta-2}{2(\theta-1)}\Big)\Big)-c^2\kappa^2\beta B\Big]\sqrt{\frac{Q}{6}} \tag{3.39}$$

Integrating factor of the above equation is:

$$e^{\int-\frac{1}{2Q}dQ}=\frac{1}{\sqrt{Q}}$$

$$\Rightarrow \qquad f(Q)\frac{1}{\sqrt{Q}}=\frac{1}{2}\Big[1-\kappa^2c^2\Big(\alpha-\beta\Big)-c^2\kappa^2\beta(B\theta)^{\frac{1}{\theta-1}}\Big(\frac{Q}{6}\Big)^{\frac{\theta-2}{2(\theta-1)}}-c^2\kappa^2\beta B\Big]\int\frac{1}{\sqrt{Q}}dQ$$

$$\Rightarrow \qquad f(Q)\frac{1}{\sqrt{Q}}=\Big[1-\kappa^2c^2(\alpha-\beta)-c^2\kappa^2\beta(B\theta)^{\frac{1}{\theta-1}}\Big(\frac{Q}{6}\Big)^{\frac{\theta-2}{2(\theta-1)}}-c^2\kappa^2\beta B\Big]\sqrt{Q}+c_5$$

$$\Rightarrow \qquad f(Q)=c_5\sqrt{Q}\Big(1-c^2\kappa^2\Big)Q-c^2\kappa^2\beta\Big(B\theta\Big)^{\frac{1}{\theta-1}}\Big(\theta-1\Big)^2\Big(\frac{Q}{6}\Big)^{\frac{\theta-2}{2(\theta-1)}}-c^2\kappa^2\beta B\sqrt{\frac{Q}{6}}lnQ \tag{3.40}$$

- **Chen-Jing model :**

The energy density for Chen-Jing model is given by;

$$\rho_D=3c^2\Big[\alpha\Big(\frac{\ddot{H}}{H}\Big)+\beta\dot{H}+\gamma H^2\Big]$$

$$\Rightarrow \qquad \rho_D=3c^2\Big[\frac{\alpha Q}{3m}-\beta B\Big(\theta\Big(\theta-1\Big)\Big)\Big(\frac{\sqrt{\frac{Q}{6}}}{B\theta}\Big)^{\frac{\theta-2}{\theta-1}}+\gamma\Big(\frac{Q}{6}\Big)\Big] \tag{3.41}$$

Now for the reconstruction we have;

$$\Rightarrow \qquad \frac{3}{\kappa^2}\Big[\frac{Q}{6}\Big(1-2f_Q\Big)+\frac{f}{6}\Big]=3c^2\Big[\frac{\alpha Q}{3m}-\beta B\Big(\theta\Big(\theta-1\Big)\Big)\Big(\frac{\sqrt{\frac{Q}{6}}}{B\theta}\Big)^{\frac{\theta-2}{\theta-1}}+\gamma\Big(\frac{Q}{6}\Big)\Big]$$

$$\Rightarrow \qquad Q\Big(1-2f_Q\Big)+f=\kappa^2c^2\Big[\frac{2\alpha}{m^2}-\beta B\Big(\Big(\theta-1\Big)\Big)\Big(\frac{1}{B\theta}\Big)^{\frac{\theta-2}{\theta-1}}+\gamma\Big]\sqrt{\frac{Q}{6}}$$

$$\Rightarrow \quad Q-2Qf_Q+f=\kappa^2c^2\Big[2\alpha-\Big(\beta B\Big(\theta-1\Big)B\theta\Big)^{\frac{2}{\theta-1}}\Big(\theta-2\Big)+\alpha^2\kappa^2c^2\Big(\frac{\theta-1}{3-\theta}\Big)+$$

$$\gamma\Big(\frac{\sqrt{Q}}{6}\Big)^{\frac{1}{\theta-1}}\Big)\Big]$$

$$\Rightarrow \qquad f_Q-\frac{1}{2Q}f=\frac{Q}{2}\Big[1-\kappa^2c^2\Big(\gamma+2\Big)\Big(-6c^2\kappa^2\alpha\Big(B\theta\Big)^{\frac{2}{\theta-1}}\Big(\theta-1\Big)$$

$$(\theta-2)\sqrt{\frac{6}{Q}}+\alpha^2c^2\kappa^2\left(\frac{Q}{6}\right)^{\frac{1}{\theta-1}}-6\beta c^2\kappa^2(B\theta)^{\frac{1}{\theta-1}}\left(\frac{Q}{6}\right)^{\frac{\theta-2}{2(\theta-1)}}\Big)$$

$$f_Q-\frac{1}{2Q}f=\frac{1}{2}\Big[1-\kappa^2c^2\Big(\gamma+2\Big)-6c^2\kappa^2\alpha\Big(B\theta\Big)^{\frac{2}{\theta-1}}\Big(\theta-1\Big)\Big(\theta-2\Big)\Big]$$

$$+\alpha c^2\kappa^2\left(\frac{Q}{6}\right)^{\frac{1}{\theta-1}}-6\beta c^2\kappa^2\Big(\theta-1\Big)\Big(B\theta\Big)^{\frac{1}{\theta-1}}\left(\frac{Q}{6}\right)^{\frac{\theta-2}{2(\theta-1)}}\Big] \tag{3.42}$$

Integrating factor of the above equation is:

$$e^{\int-\frac{1}{2Q}dQ}=\frac{1}{\sqrt{Q}}$$

Multiplying equation(3.42) the integrating factor and integrating we have:

$$\Rightarrow \qquad f(Q)\frac{1}{\sqrt{Q}}=\frac{1}{2}\Big[1-\kappa^2c^2(\gamma+2)-\Big(6c^2\kappa^2\alpha(B\theta)^{\frac{2}{\theta-1}}\sqrt{\frac{Q}{6}}(\theta-1)(\theta-2)\sqrt{\frac{6}{Q}}$$

$$+\alpha^2c^2\kappa^2\Big(\frac{Q}{6}\Big)^{\frac{1}{\theta-1}}\Big)-6\beta c^2\kappa^2\Big(\Big(B\theta\Big)^{\frac{1}{\theta-1}}\Big(\theta-1\Big)\Big(\frac{Q}{6}\Big)^{\frac{\theta-2}{2(\theta-1)}}\Big)\int\frac{dQ}{Q}\Big]Q$$

$$\Rightarrow \qquad f(Q)=c_6\sqrt{Q}+1-\kappa^2c^2\Big(\gamma+2\Big)Q-3c^2\kappa^2\alpha\Big(B\theta\Big)^{\frac{2}{\theta-1}}\Big(\theta-1\Big)\Big(\theta-2\Big)\sqrt{\frac{6}{Q}}$$

$$+12\kappa^2c^2\frac{\theta-1}{3-\theta}\Big(\frac{Q}{6}\Big)^{\frac{1}{\theta-1}}\Big)+6\beta c^2\kappa^2\Big(\Big(B\theta\Big)^{\frac{1}{\theta-1}}\Big(\theta-1\Big)^2\Big(\frac{Q}{6}\Big)^{\frac{\theta-2}{2(\theta-1)}}\Big)Q \tag{3.43}$$

3.4.4 Case IV

Considering the scale factor as [50, 51]:

$$a(t) = A\Big(B + e^{nt}\Big)^{\lambda} \tag{3.44}$$

Here we will consider A, B, n, λ as a parameters where $a < 0$ and λ will remain same.Therefore,we will solve the equation using differential equation for $f(Q)$ gravity as:

$$H = \lambda n\Big(\frac{e^{nt}}{B + e^{nt}}\Big)$$

Again for $f(Q)$ gravity we will take $Q = 6H^2$ and solve the equations:

$$H = \sqrt{\frac{Q}{6}}, \qquad \dot{H} = \frac{B}{e^{nt}}\frac{Q}{6\lambda^2 n^2}, \qquad \ddot{H} = \frac{Be^{nt}\Big(B^2 - e^{2nt}\Big)}{\Big(B + e^{nt}\Big)^2}\frac{Q}{6\lambda^2 n^2} \tag{3.45}$$

Now for the given Friedmann equation for the given $f(Q)$ gravity will be taken as:

$$3H^2 = \kappa^2\Big(\rho_m + \rho_{de}\Big)$$

- **Granda-Oliverous model:**

The energy density of Granda-Oliveros model for dark energy for exponential form of scale factor takes the form:-

$$\rho_D = 3c^2\Big(\alpha H^2 + \beta\dot{H}\Big)$$

$$\Rightarrow \qquad \rho_D = 3c^2\Big(\frac{\alpha}{6} + \beta\frac{\Big(B\theta\Big(\theta - 1\Big)\Big)}{B\theta}\Big)^{\frac{\theta-2}{\theta-1}}\sqrt{\frac{Q}{6}} \tag{3.46}$$

Comparing the energy density of $f(Q)$ gravity and Granda-Oliveros model from the equations (3.46)we have:

$$\frac{3}{\kappa^2}\Big[\frac{Q}{6}\Big(1 - 2f_a\Big) + \frac{f}{6}\Big] = 3c^2\Big[\Big(\alpha\frac{1}{n\lambda}\Big)\sqrt{\frac{Q}{6}} + \beta\left(\lambda\Big(n\lambda - \sqrt{\frac{Q}{6}}\Big)\right)^{-1} - 1\Big]$$

$$\Rightarrow \qquad Q\Big(1 - 2f_Q\Big) + f = \Big[\kappa^2 c^2\Big(\frac{\alpha}{n\lambda}\Big)\sqrt{\frac{Q}{6}} + \beta\Big(n\lambda\Big(n\lambda - \sqrt{\frac{Q}{6}}\Big)\Big)^{-1}\sqrt{\frac{Q}{6}}\Big]Q$$

$$\Rightarrow \qquad 2Qf'(Q) - f(Q) = \Big[1 - c^2\kappa^2\Big(\alpha - \frac{\beta}{\lambda}\Big) + \beta\Big(n^2\lambda^2 - \sqrt{\frac{Q}{6}}\Big)^{-1}\sqrt{\frac{Q}{6}}\Big)\Big]$$

$$\Rightarrow \qquad f'(Q) - \frac{1}{2Q}f(Q) = \frac{1}{2}\Big[1 - \kappa^2c^2\Big(\alpha - \frac{\beta}{\lambda}\Big) - c^2\kappa^2\beta nQ\Big] \tag{3.47}$$

Integrating factor of the above equation is;

$$e^{\int -\frac{1}{2Q}dQ} = \frac{1}{\sqrt{Q}}$$

Multiplying equation(3.47) and integrating factor we have:

$$\Rightarrow \qquad f(Q)\frac{1}{\sqrt{Q}} = \frac{1}{2}\Big[1 - \kappa^2c^2\Big(\alpha - \frac{\beta}{\lambda}\Big) - c^2\kappa^2\beta n\Big]\sqrt{6Q}$$

$$\Rightarrow \qquad f(Q)\frac{1}{\sqrt{Q}} = \Big[1 - \kappa^2c^2\Big(\alpha - \frac{\beta}{\lambda}\Big)Q - c^2\kappa^2\beta n\Big]\sqrt{6Q}$$

$$\Rightarrow \qquad f(Q) = c_7\sqrt{Q} + \Big[1 - \kappa^2c^2\Big(\alpha - \frac{\beta}{\lambda}\Big)Q - c^2\kappa^2\beta n\Big]\sqrt{6Q} \tag{3.48}$$

- **Chen-Jing model :**

The energy density for Chen-Jing model is given by:

$$\rho_D = 3c^2\Big[\alpha\Big(\frac{\ddot{H}}{H}\Big) + \beta\dot{H} + \gamma H^2\Big]$$

$$\Rightarrow \qquad \rho_D = 3c^2\Big[\alpha\frac{1}{\lambda n}\sqrt{\frac{Q}{6}} + \beta\Big(\frac{1}{\lambda n}\Big)\Big(\sqrt{\frac{Q}{6}}\Big) + \gamma\Big(\frac{Q}{6}\Big)\Big] \tag{3.49}$$

Now for the reconstruction we have:

$$\Rightarrow \qquad \frac{3}{\kappa^2}\Big[\frac{Q}{6}(1 - 2f_Q) + \frac{f}{6}\Big] = 3c^2\Big[\frac{\alpha 6\lambda^2n^2\frac{1}{n\lambda}\sqrt{\frac{Q}{6}}}{\sqrt{\frac{Q}{6}}} + \beta 6\lambda^2n^2B + \gamma\frac{Q}{6}\Big]$$

$$\Rightarrow \qquad Q(1 - 2f_Q) + f = 1 - \kappa^2c^2\Big[\frac{12\lambda^2n^3e^{2nt}(B + e^{nt})^2 - e^{3nt}(B + e^{nt})}{(B + e^{nt})^4} + \kappa^2c^2n\sqrt{6Q}$$

$$+\Big(\frac{5\alpha}{\lambda} - 4\alpha + \beta\Big)\Big]$$

$$\Rightarrow \qquad Q\Big(1 - 2f_Q\Big) + f = \frac{1}{2}\Big[1 - \kappa^2c^2\Big(\frac{4\alpha}{\lambda} - \frac{2\alpha}{\lambda^2} - \frac{\beta}{\lambda} + \gamma\Big)$$

$$+18n^2\alpha\kappa^2c^2-\kappa^2c^2\sqrt{6Q}+\kappa^2c^2n\Big(\frac{5\alpha}{\lambda}-4\alpha+\beta\Big)\sqrt{6Q} \tag{3.50}$$

Integrating factor of the above equation is:

$$e^{\int-\frac{1}{2Q}dQ}=\frac{1}{\sqrt{Q}}$$

Multiplying equation(3.50) the integrating factor and integrating we have:

$$f(Q)\frac{1}{\sqrt{Q}}=\frac{Q}{2}\Big[1-\kappa^2c^2\Big(\frac{4\alpha}{\lambda}-\frac{2\alpha}{\lambda^2}-\frac{\beta}{\lambda}+\gamma\Big)+18n^2\alpha^2\kappa^2c^2-\kappa^2c^2n^2\sqrt{6Q}\Big(\frac{5\alpha}{\lambda}-4\alpha+\beta\Big)\Big]$$

$$\Rightarrow \qquad f(Q)\frac{1}{\sqrt{Q}}=\Big[1-\kappa^2c^2\Big(\frac{4\alpha}{\lambda}-\frac{2\alpha}{\lambda^2}-\frac{\beta}{\lambda}+\gamma\Big)+18n^2\alpha^2\kappa^2c^2-\kappa^2c^2\Big(\frac{5\alpha}{\lambda}-4\alpha+\beta\Big)\Big]$$

$$\Rightarrow \qquad f(Q)=c_8\sqrt{Q}\Big[1-\kappa^2c^2\Big(\frac{4\alpha}{\lambda}-\frac{2\alpha}{\lambda^2}-\frac{\beta}{\lambda}+\gamma\Big)+18n^2\alpha^2\kappa^2c^2-\kappa^2c^2\Big(\frac{5\alpha}{\lambda}-4\alpha+\beta\Big)\Big]lnQ \tag{3.51}$$

3.5 Energy Condition:

Energy conditions play an important role in the presence of matter which can be determined using energy condition. The requirements only consider the geodesic structure of spacetime. Hence the energy condition leads to many singularity theorem, from the General Theory of Relativity. Now we will considered the definition and formation of many modified theory of gravities. In particular, with the degree of freedom of modified gravities of modified gravity gives results if it combined with the energy conditions that provide the gravity of non-attractive nature. The significant role of cosmological model is free from all the cosmological problems and have the consistent observation. Hence the conclusion of each and every matter reveal the thermodynamic properties and the gravity will continue to attract the high negative pressure. Henceforth, the normal matter will have repulsive nature gravity. Now we can put together all the consistent energy conditions in a wide range of theories since the additional degree of freedom will leads to different modified gravities that will fall under different sections under the group of many effective fluids. These are the important factor from the cosmological point of view. Hence we observed the existence of dark energy from the

General Relativity approach. Therefore the energy condition might be change after the lot of effects on different modified gravities.

In this segment we will consider on $f(Q)$ theory of gravity. We will observe the parameter that is having the limitations on the graphical representation. Here we will discuss four types of energy conditions with their certain limitation based on the model Granda-Olivers and Chen-Jing. We will focus on GO model with their graphical representation on the following different cases.

(I) Weak Energy Condition(WEC) $\Rightarrow$ $\rho_{eff} \geq 0, \quad \rho_{eff} + p_{eff} \geq 0$

(II) Null Energy Condition(NEC) $\Rightarrow$ $\rho_{eff} + p_{eff} \geq 0$

(III) Dominant Energy Condition(DEC)$\Rightarrow$ $\rho_{eff} \geq 0, \quad \rho_{eff} \geq |p_{eff}|$

(IV)Strong Energy Condition(SEC) $\Rightarrow$ $\rho_{eff} + 3p_{eff} \geq 0$

$$\rho_{eff} = \frac{1}{f'(Q)}\Big(\rho + \frac{f}{2}\Big) \tag{3.52}$$

$$p_{eff} = -2\frac{f'(\dot{Q})}{f'(Q)}H + \frac{1}{f'(Q)}\Big(p + \frac{f}{2}\Big) \tag{3.53}$$

- **Granda-Oliverous Model :**

3.5.1 Case I :

Considering the power law form as $a(t) = a_0 t^m$.

Here we will explain the energy condition for the first case and we will get the inequalities WEC, NEC, DEC, SEC for the GO models where c, c_1, c_2, $\kappa = 1$:

From equation(3.22)

$$f(Q) = c_1\sqrt{Q} + \Big[1 - \frac{\kappa^2 c^2}{m}\Big(\alpha m - \beta\Big)\Big]Q$$

$$f'(Q) = \frac{c_1}{2\sqrt{Q}} + \Big[1 - \frac{\kappa^2 c^2}{m}\Big(\alpha m - \beta\Big)\Big]$$

Now,

$$f'(Q) = f'(Q).\dot{Q} = -\frac{c_1}{2\sqrt{6}m}$$

Again,

$$\rho_{de} = \frac{3}{\kappa^2}\Big[\frac{Q}{6}\Big(1-2f_a\Big)+\frac{f}{6}\Big]$$

$$\rho_{de} = \frac{1}{2\kappa^2}\Big[Q(1-2f_a)+f\Big]$$

$$= \frac{1}{2\kappa^2}\Big[Q\Big[1-\frac{c_1}{\sqrt{Q}}-2+\frac{2c^2\kappa^2}{m}\Big(\alpha m-\beta\Big)\Big]+c_1\sqrt{Q}+1-\frac{c^2\kappa^2}{m}\Big(\alpha m-\beta\Big)Q\Big]$$

$$= \frac{1}{2\kappa^2}\Big[Q\Big[-1-\frac{c_1}{\sqrt{Q}}-2+\frac{2c^2\kappa^2}{m}\Big(\alpha m-\beta\Big)\Big]+c_1\sqrt{Q}+1-\frac{c^2\kappa^2}{m}\Big(\alpha m-\beta\Big)Q\Big]$$

$$= \frac{1}{2\kappa^2}\Big[-Q-c_1\sqrt{Q}+\frac{2c^2\kappa^2}{m}\Big(\alpha m-\beta\Big)Q+c_1\sqrt{Q}-\frac{c^2\kappa^2}{m}\Big(\alpha m-\beta\Big)Q\Big]$$

$$= \frac{1}{2\kappa^2}\frac{c^2\kappa^2}{m}\Big(\alpha m-\beta\Big)Q$$

$$\rho_{de} = \frac{c^2}{2m}\Big(\alpha m-\beta\Big) \tag{3.54}$$

Again,

$$\rho_{de} = -\frac{1}{\kappa^2}\Big[2\dot{H}\Big(1-2f_Q\Big)+\frac{f}{2}+3H^2\Big(1-8f_{QQ}\dot{H}-2f_Q\Big)\Big]$$

$$= -\frac{1}{\kappa^2}\Big[\frac{c^{\kappa 2}}{m^2}\Big(\alpha m-\beta\Big)Q-\frac{c^2\kappa^2}{3m^2}\Big(\alpha m-\beta\Big)\Big]$$

$$= -\frac{1}{\kappa^2}\Big[\frac{c^2\kappa^2}{m^2}\Big(\alpha m-\beta\Big)Q\Big(\frac{m}{2}-\frac{1}{3}\Big)\Big]$$

$$\rho_{de} = -\frac{c^2}{6m^2}\Big(3m-2\Big)\Big(\alpha m-\beta\Big)Q \tag{3.55}$$

Now,

$$\rho_{eff} = \frac{1}{f_Q}\Big(\rho+\frac{f}{2}\Big)$$

$$= \frac{1}{f_Q}\Big(\rho_{de}+\frac{f}{2}\Big)$$

$$= \frac{\frac{c^2}{2m}\Big(\alpha m-\beta\Big)Q+\frac{c_1\sqrt{Q}}{2}+\Big[1-\frac{c^2\kappa^2}{m}\Big(\alpha m-\beta\Big)\Big]\frac{Q}{2}}{1+\frac{c_1}{2\sqrt{Q}}-\frac{c^2\kappa^2}{m}\Big(\alpha m-\beta\Big)}$$

$$= \frac{\frac{Q}{2}+\frac{c_1}{2}+\frac{c^2}{2m}\Big(1-\kappa^2\Big)\Big(\alpha m-\beta\Big)Q}{1+\frac{c_1}{2\sqrt{Q}}-\frac{c^2\kappa^2}{m}\Big(\alpha m-\beta\Big)} \tag{3.56}$$

And,

$$p_{eff} = -2\frac{f(\dot{Q})}{f'(Q)}H+\frac{1}{f'(Q)}\Big(p+\frac{f}{2}\Big)$$

$$= -2\frac{f''(Q)}{f'(Q)}\dot{Q}H+\frac{1}{f'(Q)}\Big(p_{de}+\frac{f}{2}\Big)$$

$$= -2\frac{f''(Q)}{f'(Q)}(12H\dot{H})H+\frac{1}{f'(Q)}\Big(p_{de}+\frac{f}{2}\Big)$$

$$= -24\frac{f''(Q)}{f'(Q)}H^2\dot{H}+\frac{1}{f'(Q)}\Big(p_{de}+\frac{f}{2}\Big)$$

$$= -24\frac{f''(Q)}{f'(Q)}\Big(\frac{Q}{6}\Big)\Big(\frac{-Q}{6m}\Big)+\frac{1}{f'(Q)}\Big(p_{de}+\frac{f}{2}\Big)$$

Therefore,

$$p_{eff} = \frac{2Q^2}{3m}\frac{f''(Q)}{f'(Q)}+\frac{1}{f'(Q)}\Big(\rho_{de}+\frac{f}{2}\Big)$$

$$= -\frac{\frac{c\sqrt{Q}}{6m}-\frac{c^2}{6m}\Big(3m-2\Big)\Big(\alpha m-\beta\Big)Q+\frac{c_1\sqrt{Q}}{2}+\frac{1}{2}\Big[1-\frac{c^2\kappa^2}{2m}\Big(\alpha m-\beta\Big)\Big]Q}{1+\frac{c^2\kappa^2}{2\sqrt{Q}}-\frac{c^2\kappa^2}{m}\Big(\alpha m-\beta\Big)} \tag{3.57}$$

Therefore,

$$p_{eff}+\rho_{eff} = \frac{\frac{c^2}{3m^2}\Big(\alpha m-\beta\Big)Q+\Big(1-\frac{1}{6m}\Big)c_1\sqrt{Q}+\Big[1-\frac{c^2\kappa^2}{m}\Big(\alpha m-\beta\Big)\Big]Q}{1-\frac{c_1}{2\sqrt{Q}}-\frac{c^2\kappa^2}{m}\Big(\alpha m-\beta\Big)}$$

For WEC:

$$\Rightarrow \qquad \frac{\frac{Q}{2}+\frac{c_1\sqrt{Q}}{2}+\frac{c^2}{2m}\Big(1-\kappa^2\Big)\Big(\alpha m-\beta\Big)Q}{1-\frac{c_1}{2\sqrt{Q}}-\frac{c^2\kappa^2}{m}\Big(\alpha m-\beta\Big)} \geq 0 \tag{3.58}$$

and,

$$\frac{\frac{c^2}{3m}\Big(\alpha m-\beta\Big)Q+\Big(1-\frac{1}{6m}\Big)c_1\sqrt{Q}+\Big[1-\frac{c^2\kappa^2}{m}\Big(\alpha m-\beta\Big)\Big]Q}{1-\frac{c_1}{2\sqrt{Q}}-\frac{c^2\kappa^2}{m}\Big(\alpha m-\beta\Big)}\geq 0 \tag{3.59}$$

For NEC:

$$\frac{-Q\Big[c_1\Big(m-6m^2\Big)+2\Big(3m-1\Big)\sqrt{Q}-c^2\beta\kappa^2+m\Big(c^2\alpha\kappa^2-1\Big)\Big]}{3m\Big(c_1m+2\sqrt{Q}(m-c^2\alpha\kappa^2+c^2\beta\kappa^2\Big)}\geq 0 \tag{3.60}$$

For DEC:

$$\frac{c_1\Big(\sqrt{Q}-\frac{1}{2\sqrt{Q}}\Big)}{1+\frac{c_1}{2\sqrt{Q}}-\frac{c^2\kappa^2\Big(\alpha m-\beta\Big)}{m}}\geq 0 \tag{3.61}$$

$$\frac{-c_1mQ}{c_1m+2\sqrt{Q}\Big(m-c^2m\alpha\kappa^2+c^2\beta\kappa^2\Big)}+|Q-\frac{Q}{3m}|\leq 0 \tag{3.62}$$

For SEC:

$$\frac{-Q\Big[c_1\Big(m-4m^2\Big)+2\Big(-1-3m\Big)\sqrt{Q}-c^2\beta\kappa^2+m\Big(-1+c^2\alpha\kappa^2\Big)\Big]}{m\Big[c_1+2\sqrt{Q}\Big(m-c^2\alpha\kappa^2+c^2\beta\kappa^2\Big)\Big]} \tag{3.63}$$

3.5.2 Case II:

Considering the scale factor $a(t)$ as $a(t)=a_0(t_s-t)^-n$.

Here,we will explain the energy condition for the first case and we will get the inequalities WEC, NEC, DEC, SEC for the GO models where c, c_1, c_2, $\kappa=1$:

From equation (3.31)

$$f(Q)=1-\frac{c^2\kappa^2}{n}\Big(\alpha n+\beta\Big)Q$$

$$\Rightarrow \qquad f'(Q)=\frac{c_2}{2\sqrt{Q}}+1-\frac{c^2\kappa^2}{n}\Big(\alpha n+\beta\Big)$$

$$f'(Q)=f'(Q).\dot{Q}=-\frac{c_2}{2\sqrt{6}n}$$

Again,

$$\rho_{de}=\frac{3}{\kappa^2}\Big[\frac{Q}{6}\Big(1-2f_Q\Big)+\frac{f}{6}\Big]$$

$$\rho_{de} = \frac{1}{2\kappa^2}\Big[Q(1-2f_Q)+f\Big]$$

$$= \frac{1}{2\kappa^2}\Big[Q\Big(1-\frac{c_2}{\sqrt{Q}}-2+\frac{2c^2\kappa^2}{n}\Big(\alpha n+\beta\Big)\Big)\Big]+c_2\sqrt{Q}+\Big[1-\frac{c^2\kappa^2}{n}\Big(\alpha n+\beta\Big)\Big]Q$$

$$= \frac{1}{2\kappa^2}\Big[Q\Big(1-\frac{c_2}{\sqrt{Q}}+\frac{2c^2\kappa^2}{n}\Big(\alpha n+\beta\Big)\Big)\Big]+c_2\sqrt{Q}+\Big[1-\frac{c^2\kappa^2}{n}\Big(\alpha n+\beta\Big)\Big]Q$$

$$= \frac{1}{2\kappa^2}\Big[-Q-\frac{c_2}{\sqrt{Q}}+\frac{2c^2\kappa^2}{n}\Big(\alpha n+\beta\Big)\Big)\Big]Q+c_2\sqrt{Q}+Q-\frac{c^2\kappa^2}{n}\Big(\alpha n+\beta\Big)Q\Big]$$

$$= \frac{1}{2\kappa^2}\frac{c^2\kappa^2}{n}\Big(\alpha n+\beta\Big)Q$$

$$\rho_{de} = \frac{c^2}{2n}\Big(\alpha n+\beta\Big)Q \tag{3.64}$$

Again,

$$\rho_{de} = -\frac{1}{\kappa^2}\Big[2\dot{H}\Big(1-2f_Q\Big)+\frac{f}{2}+3H^2\Big(1-8f_{QQ}\dot{H}-2f_Q\Big)\Big]$$

$$= -\frac{1}{\kappa^2}\Big[\Big(\frac{-Q}{3n}\Big)\Big(-\frac{c_2}{\sqrt{Q}}+\frac{c^2\kappa^2}{n}\Big(\alpha n+\beta\Big)\Big)\Big]+\frac{c_2\sqrt{Q}}{2}+\frac{1}{2}\Big[1-\frac{c^2\kappa^2}{n}\Big(\alpha n+\beta\Big)Q\Big]$$

$$+\frac{Q}{2}\Big[1+\frac{2c_2Q^2}{3Q\sqrt{Q}}\Big]-\frac{c_2}{\sqrt{Q}}-2+\frac{2c^2\kappa^2}{n}\Big(\alpha n+\beta\Big)$$

$$= -\frac{1}{\kappa^2}\Big[\frac{c^2\kappa^2}{2n}\Big(\alpha n+\beta\Big)Q-\frac{c^2\kappa^2}{3n^2}\Big(\alpha n+\beta\Big)\Big]$$

$$= -\frac{1}{\kappa^2}\Big[\frac{c^2\kappa^2}{n^2}\Big(\alpha n+\beta\Big)Q\Big(\frac{n}{2}+\frac{1}{3}\Big)\Big]$$

$$\rho_{de} = \frac{c^2}{6n^2}\Big(3n+2\Big)\Big(\alpha n+\beta\Big)Q \tag{3.65}$$

Again,

$$\rho_{de} = 3c^2\Big[\alpha\Big(\frac{\ddot{H}}{H}\Big)+\beta\dot{H}+\gamma H^2\Big]$$

$$= 3c^2\Big[\alpha\Big(\frac{Q}{3n^2}\Big)+\frac{\beta Q}{6n}+\frac{\gamma Q}{6}\Big]$$

$$= c^2\kappa^2\Big[\frac{2\alpha}{n^2} + \frac{\beta}{n} + \gamma\Big]Q$$

$$= \frac{c^2\kappa^2}{n^2}\Big[2\alpha + n\beta + \gamma n^2\Big]Q$$

$$= \Big[1 - \frac{c^2\kappa^2}{n^2}\Big(2\alpha + n\beta + \gamma n^2\Big)\Big]Q$$

$$= \frac{1}{2}\Big[1 - \frac{c^2\kappa^2}{n^2}\Big(2\alpha + n\beta + \gamma n^2\Big)\Big]\sqrt{Q} + c_3$$

Now,

$$\rho_{eff} = \frac{1}{f_Q}\Big(\rho + \frac{f}{2}\Big)$$

$$= \frac{1}{f_Q}\Big(\rho_{de} + \frac{f}{2}\Big)$$

$$= \frac{\frac{c^2}{2n^2}\Big(\alpha n + \beta\Big) + \frac{c_3\sqrt{Q}}{2} + \frac{Q}{2} - \frac{c^2\kappa^2}{2n}\Big(\alpha n + \beta\Big)Q}{1 + \frac{c_3}{2\sqrt{Q}} - \frac{c^2\kappa^2}{n}\Big(2\alpha + n\beta + \gamma n^2\Big)}$$

$$= \frac{\frac{Q}{2} + \frac{c_3\sqrt{Q}}{2} + \frac{c^2}{2n}\Big[1 - \frac{c^2\kappa^2}{n^2}\Big(2\alpha + n\beta + n^2\gamma\Big)\Big]}{\Big[1 + \frac{c_3}{2\sqrt{Q}} - \frac{c^2\kappa^2}{n}\Big(\alpha n + \beta\Big)\Big]} \tag{3.66}$$

And,

$$p_{eff} = -2\frac{f(\dot{Q})}{f'(Q)}H + \frac{1}{f'(Q)}\Big(p + \frac{f}{2}\Big)$$

$$= -2\frac{f''(Q)}{f'(Q)}\dot{Q}H + \frac{1}{f'(Q)}\Big(p_{de} + \frac{f}{2}\Big)$$

$$= -2\frac{f''(Q)}{f'(Q)}(12H\dot{H})H + \frac{1}{f'(Q)}\Big(p_{de} + \frac{f}{2}\Big)$$

$$= -24\frac{f''(Q)}{f'(Q)}H^2\dot{H} + \frac{1}{f'(Q)}\Big(p_{de} + \frac{f}{2}\Big)$$

$$= -24\frac{f''(Q)}{f'(Q)}\Big(\frac{Q}{6}\Big)\Big(\frac{-Q}{6n}\Big) + \frac{1}{f'(Q)}\Big(p_{de} + \frac{f}{2}\Big)$$

Now,

$$p_{eff} = \frac{2Q^2}{3n}\frac{f''(Q)}{f'(Q)} + \frac{1}{f'(Q)}\Big(\rho_{de} + \frac{f}{2}\Big)$$

$$= \frac{\frac{c_3\sqrt{Q}}{6n} - \frac{Q^2}{6n}\Big(3n+1\Big)\Big(\alpha n+\beta\Big) + \frac{c_3\sqrt{Q}}{2} + \frac{1}{2}\Big[1-\kappa^2c^2\Big(2\alpha+n\beta+n^2\gamma\Big)\Big]Q}{1+\frac{c^2\kappa^2}{2}\sqrt{Q} - \frac{c^2\kappa^2}{n}\Big(\alpha n+\beta\Big)} \tag{3.67}$$

Therefore,

$$\rho_{eff} + p_{eff} = \frac{\frac{c^2}{3n^2}\Big(\alpha n+\beta\Big)Q + \Big(1+\frac{1}{6n}\Big)c_3\sqrt{Q} + \Big[1-\frac{c^2\kappa^2}{n}\Big(2\alpha+n\beta+\gamma n^2\Big)\Big]}{1+\frac{c^2\kappa^2}{2\sqrt{Q}} - \frac{c^2\kappa^2}{n}\Big(\alpha n+\beta\Big)}$$

For WEC:

$$= \frac{c_3nQ}{c_3 + 2\sqrt{Q}\Big(n - c^2\alpha\kappa^2 + c^2\beta\kappa\Big)} \tag{3.68}$$

$$= \frac{Q\Big[-c_3\Big(1+6n\Big) + 2\Big(1+3n\Big)\sqrt{Q} - c^2\kappa^2\beta + n\Big(-1+c^2\kappa^2\Big)\Big]}{3n\Big[c_3n + 2\sqrt{Q}\Big(n^2 - c^2n\alpha\kappa^2 + c^2\kappa^2\beta\Big)\Big]} \tag{3.69}$$

For NEC:

$$\frac{-Q\Big[c_3\Big(1+6n\Big) + 2\Big(3n+1\Big)\sqrt{Q} - c^2\beta\kappa^2 + n\Big(c^2\alpha\kappa^2 - 1\Big)\Big]}{3n\Big(c_3n + 2\sqrt{Q}(n - c^2\alpha\kappa^2 + c^2\beta\kappa^2\Big)} \geq 0 \tag{3.70}$$

For DEC:

$$\frac{c_3\Big(\sqrt{Q} - \frac{1}{2\sqrt{Q}}\Big)}{1+\frac{c_3}{2\sqrt{Q}} - \frac{c^2\kappa^2\Big(\alpha n-\beta\Big)}{n}} \geq 0 \tag{3.71}$$

$$\frac{-c_3nQ}{c_3n + 2\sqrt{Q}\Big(n - c^2n\alpha\kappa^2 + c^2\beta\kappa^2\Big)} + |Q - \frac{Q}{3n}| \leq 0 \tag{3.72}$$

For SEC:

$$\frac{-Q\Big[c_3\Big(1+4n\Big) + 2\Big(1+3n\Big)\sqrt{Q} - c^2\beta\kappa^2 + n\Big(-1+c^2\alpha\kappa^2\Big)\Big]}{n\Big[c_1 + 2\sqrt{Q}\Big(n - c^2\alpha\kappa^2 + c^2\beta\kappa^2\Big)\Big]} \geq 0 \tag{3.73}$$

3.5.3 Case III:

Considering the scale factor $a(t) = e^{Bt^\theta}$.

Here we will solve the energy condition and we will get the inequalities WEC, NEC,

DEC, SEC, for the GO models:

From equation 3.40

$$f(Q) = \Big[1 - c^2\kappa^2\Big(\alpha - \beta\Big)Q - c^2\kappa^2\Big(B\theta\Big)^{\frac{1}{\theta-1}}\Big(\theta - 1\Big)^{\frac{\theta-2}{2(\theta-1)}}c^2\kappa^2\beta B\sqrt{\frac{Q}{6}}\Big]$$

$$f'(Q) = \Big[1 - c^2\kappa^2(\alpha - \beta) - \frac{c^2\kappa^2\beta B}{12\sqrt{Q}}\Big]$$

Now,

$$f'(Q) = f''(Q)\dot{Q} = \frac{c^2\kappa^2\beta B}{12Q^{\frac{3}{2}}n}\sqrt{\frac{Q}{6}}$$

Again,

$$\rho_{de} = \frac{3}{\kappa^2}\Big[\frac{Q}{6}\Big(1 - 2f_Q\Big) + \frac{f}{6}\Big)\Big]$$

$$\rho_{de} = \frac{1}{2\kappa^2}\Big[Q(1 - 2f_Q) + f\Big]$$

$$= \frac{1}{2\kappa^2}\Big[Q - 2\Big(1 - c^2\kappa^2\Big(\alpha - \beta\Big)Q\Big) - c^2\kappa^2\Big(B\theta\Big)^{\frac{1}{\theta-1}}\Big(\theta - 1\Big)^{\frac{\theta-2}{2(\theta-1)}}$$

$$-c^2\kappa^2\beta B\sqrt{\frac{Q}{6}} + 1 - c^2\kappa^2\Big(\alpha - \beta\Big)Q\Big]$$

$$= \Big[1 - c^2\kappa^2\Big(\alpha - \beta\Big) - c^2\kappa^2\beta\Big(B\theta\Big)^{\frac{1}{\theta-1}}\Big(\frac{Q}{6}\Big)^{\frac{\theta-2}{2(\theta-1)}} - c^2\kappa^2\beta B\Big]\sqrt{\frac{Q}{6}}$$

$$= \frac{1}{2}\Big[1 - c^2\kappa^2\Big(\alpha - \beta\Big) - c^2\kappa^2\beta\Big(B\theta\Big)^{\frac{1}{\theta-1}}\Big(\frac{Q}{6}\Big)^{\frac{\theta-2}{2(\theta-1)}} - c^2\kappa^2\beta B\Big]\int\frac{dQ}{\sqrt{Q}}$$

$$= \Big[1 - c^2\kappa^2\Big(\alpha - \beta\Big) - c^2\kappa^2\beta\Big(B\theta\Big)^{\frac{1}{\theta-1}}\Big(\frac{Q}{6}\Big)^{\frac{\theta-2}{2(\theta-1)}} - c^2\kappa^2\beta B\Big]\sqrt{\frac{Q}{6}}lnQ + c_5$$

Again,

$$\rho_{de} = -\frac{1}{\kappa^2}\Big[2\dot{H}\Big(1 - 2f_Q\Big) + \frac{f}{2} + 3H^2\Big(1 - 8f_{QQ}\dot{H} - 2f_Q\Big)\Big]$$

$$= -\frac{-1}{\kappa^2}\Big[2B\theta^2 - 2B\theta\Big(\frac{\sqrt{\frac{Q}{6}}}{B\theta}\Big)^{\frac{\theta-2}{\theta-1}}1 - c^2\kappa^2\Big(\alpha - \beta\Big) - c^2\kappa^2\beta\Big(B\theta\Big)^{\frac{1}{\theta-1}}\frac{\theta - 2}{2(\theta - 1)} - c^2\kappa^2\beta B\Big]$$

$$\sqrt{\frac{Q}{6}}+\frac{c^2\kappa^\beta B\theta}{12Q^{\frac{3}{2}}\sqrt{\frac{Q}{6}}}+\frac{Q}{2}\Big[1-c^2\kappa^2\Big(\alpha-\beta\Big)-c^2\kappa^2\beta\Big(B\theta\Big)^{\frac{1}{\theta-1}}\Big(\frac{Q}{6}\Big)^{\frac{\theta-2}{2(\theta-1)}}-c^2\kappa^2\beta B\Big]$$

$$B\theta\Big(\theta-1\Big)t^(\theta-2)-2c_5\sqrt{Q}1+c^{\kappa^2}\Big(\alpha-\beta\Big)Q-c^2\kappa^2\beta\Big(B\theta\Big)^{\frac{1}{\theta-1}}\Big(\theta-1\Big)^2\Big] \tag{3.74}$$

Again,

$$\rho_{eff}=\frac{1}{f_Q}\Big(\rho+\frac{f}{2}\Big)$$

$$=\frac{1}{f_Q}\Big(\rho_{de}+\frac{f}{2}\Big)$$

$$=\Big[1-c^2\kappa^2\Big(\alpha-\beta\Big)-c^2\kappa^2\beta\Big(B\theta\Big)^{\frac{1}{\theta-1}}\Big(\frac{Q}{6}\Big)^{\frac{\theta-2}{2(\theta-1)}}-c^2\kappa^2\beta B\Big]B\theta\Big(\theta-1\Big)t^(\theta-2)$$

$$-2c_5\sqrt{Q}1+c^{\kappa^2}\Big(\alpha-\beta\Big)Q-c^2\kappa^2\beta\Big(B\theta\Big)^{\frac{1}{\theta-1}}\Big(\theta-1\Big)^2\Big]6^{(1-\frac{3\theta}{2})}c_5\sqrt{Q}-2^{\frac{3+\theta^2}{2}}$$

$$+6^{\frac{3\theta}{2}}c^2Q^{\frac{1}{2}}(-2+\theta)(-1+\theta)-6^{\frac{3-3\theta+\theta^2}{2}}B\sqrt{Q}\beta n^2c^2\kappa^2(B\theta)^{\frac{1}{\theta-1}}B\sqrt{Q}\beta nc^2\kappa^2$$

$$\Big[\Big(\alpha-\beta\Big)Q-c^2\kappa^2\beta\Big(B\theta\Big)^{\frac{1}{\theta-1}}\Big(\theta-1\Big)^2\Big]6^{1-\frac{3\theta}{2}}c_5\sqrt{Q}-2^{\frac{3+\theta^2}{2}}B\sqrt{Q}\beta nc^2\kappa^2$$

$$+6^{\frac{3\theta}{2}}c^2Q^{\frac{1}{2}(-2+\theta)}-6^{\frac{3-3\theta+\theta^2}{2}}B\sqrt{Q}\beta n^2c^2\kappa^2(B\theta)^{\frac{1}{\theta-1}}\Big]^{-1} \tag{3.75}$$

Now,

$$p_{eff}=-2\frac{f(\dot{Q})}{f'(Q)}H+\frac{1}{f'(Q)}\Big(p+\frac{f}{2}\Big)$$

$$=-2\frac{f''(Q)}{f'(Q)}\dot{Q}H+\frac{1}{f'(Q)}\Big(p_{de}+\frac{f}{2}\Big)$$

$$=-2\frac{f''(Q)}{f'(Q)}(12H\dot{H})H+\frac{1}{f'(Q)}\Big(p_{de}+\frac{f}{2}\Big)$$

$$=-24\frac{f''(Q)}{f'(Q)}H^2\dot{H}+\frac{1}{f'(Q)}\Big(p_{de}+\frac{f}{2}\Big)$$

$$= -24\frac{f''(Q)}{f'(Q)}\Big(\frac{Q}{6}\Big)\Big(\frac{-Q}{6n}\Big) + \frac{1}{f'(Q)}\Big(p_{de} + \frac{f}{2}\Big)$$

$$p_{eff} = \frac{2Q^2}{3n}\frac{f''(Q)}{f'(Q)} + \frac{1}{f'(Q)}\Big(\rho_{de} + \frac{f}{2}\Big)$$

$$= \frac{2Q^2}{3n}\Big(-\frac{c^2\kappa^2\beta B}{24Q^{\frac{3}{2}}}\Big) - \frac{c^2}{6n^2}\Big(\frac{c^2\kappa^2\beta BQ}{12Q^{\frac{3}{2}}}\Big)\sqrt{\frac{Q}{6}} + \frac{f}{2}$$

$$\Big[1 - c^2\kappa^2\Big(\alpha - \beta\Big) - \frac{c^2\kappa^2\beta B}{12\sqrt{Q}}\Big]^{-1} \tag{3.76}$$

For WEC:

$$\Big[Q\Big(6^{1-\frac{3\theta}{2}}\Big)\Big(6^{1+\frac{\theta^2}{2}}\Big)c_5\sqrt{Q} + Q^{\frac{1}{2}-\frac{3\theta}{2}}\Big(B\theta\Big)^{\frac{1}{-1+\theta}} - 6^{\frac{3\theta}{2}}c^2Q^{\frac{1+\theta^2}{2}}\beta\Big(-3+\theta\Big)\Big(-1+\theta\Big)^2$$

$$\theta + 2^{\frac{2+\theta^2}{2}}3^{\frac{1+\theta^2}{2}}BQ^{\frac{3\theta}{2}}\beta nc^2\Big(B\theta\Big)^{\frac{1}{1-\theta}}\kappa^2 - 6^{\frac{3-3\theta+\theta^2}{2}}B\sqrt{Q}\beta nc^2\kappa^2 lnQ\Big]$$

$$\Big[6^{1-\frac{3\theta}{2}}6^{1+\frac{\theta^2}{2}}c_5\sqrt{Q} - 2^{\frac{3+\theta^2}{2}}3^{\frac{1+\theta^2}{2}}B\sqrt{Q}\beta nc^2\kappa^2 + 6^{\frac{3\theta}{2}}c^2Q^{\frac{1}{2}(-2+\theta)(-1+\theta)}$$

$$\beta\Big(-2+\theta\Big)\Big(-1+\theta\Big)^3\Big(B\theta\Big)^{\frac{1}{\theta-1}}\kappa^2 - 2^2 + \frac{\theta^2}{2}3^{1+\frac{\theta^2}{2}}Q\Big(-1 + c^2\Big(\alpha-\beta\Big)\kappa^2\Big)$$

$$-6^{\frac{3-3\theta+\theta^2}{2}}B\sqrt{Q}\beta nc^2\kappa^2 lnQ\Big]^{-1} \geq 0 \tag{3.77}$$

And,

$$\Big[2Q(6c_5 + 12\sqrt{Q} - 12c^2\sqrt{Q}\Big(\alpha-\beta\Big)\kappa^2 - 2\sqrt{6}B\beta nc^2\kappa^2 + 6^{\frac{-(-3+\theta)\theta}{2}}\beta\Big(-2+\theta\Big)$$

$$\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\kappa^2 - \sqrt{6}B\beta nc^2\kappa^2 lnQ) - 6^{\frac{\theta}{2(-1+\theta)}}B\Big(\frac{\sqrt{Q}}{6}\Big)^{\frac{-2+\theta}{-1+\theta}}\theta\Big(1 - \Big(1 + 6^{\frac{\theta}{2-2\theta}}B\Big(\frac{\sqrt{Q}}{6}\Big)^{\frac{\theta}{-1+\theta}})\theta$$

$$\Big(6c_5 + 12\sqrt{Q} - 12c^2\sqrt{Q}\Big(\alpha-\beta\Big)\kappa^2 - 2\sqrt{Q}B\beta nc^2\kappa^2 + 6^{\frac{-(-3+\theta)\theta}{2}}c^2Q^{\frac{1-3\theta+\theta^2}{2}}$$

$$\beta\Big(-2+\theta\Big)\Big(-1+\theta\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\kappa^2 - \sqrt{6}Bnc^2\kappa^2 lnQ\Big) + 2^{\frac{-(-2+\theta)(-1+\theta)}{2}}3^{\frac{-(-3+\theta)\theta}{2}}$$

$$6^{1-\frac{3\theta}{2}}Q^{1-\frac{3\theta}{2}}\Big(6^{1+\frac{\theta^2}{2}}c_5Q^{\frac{3\theta}{2}} + \Big(B\theta\Big)^{\frac{1}{-1+\theta}}\Big(-6^{\frac{3\theta}{2}}c^2Q^{\frac{1+\theta^2}{2}}\beta\Big(-3+\theta\Big)\Big(-1+\theta\Big)^2\theta$$

$$+2^{\frac{3+\theta^2}{2}}3^{\frac{1+\theta^2}{2}}BQ^{\frac{3\theta}{2}}\beta nc^2\Big(B\theta\Big)^{\frac{1}{\theta-1}}\Big)\kappa^2\Big)-6^{\frac{3-3\theta+\theta^2}{2}}BQ\beta nc^2\kappa^2 lnQ)\Big]\Big[3\sqrt{Q}\Big(12+\frac{6c_5}{\sqrt{Q}}$$

$$-12c^2\Big(\alpha-\beta\Big)\kappa^2-\frac{2\sqrt{6}B\beta nc^2\kappa^2}{\sqrt{Q}}+6^{\frac{-(-3+\theta)}{2}}c^2Q^{\frac{-(-3+\theta)}{2}}\beta\Big(-2+\theta\Big)\Big(-1+\theta\Big)^3$$

$$\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\kappa^2-\frac{\sqrt{6}B\beta nc^2\kappa^2 lnQ}{\sqrt{Q}}\Big]^{-1}\geq 0 \tag{3.78}$$

For NEC:

$$\Big[2Q(6c_5+12\sqrt{Q}-12c^2\sqrt{Q}\Big(\alpha-\beta\Big)\kappa^2-2\sqrt{6}B\beta nc^2\kappa^2+6^{\frac{-(-3+\theta)\theta}{2}}\beta\Big(-2+\theta\Big)$$

$$\Big(-1+\theta\Big)^3\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\kappa^2-\sqrt{6}B\beta nc^2\kappa^2 lnQ)-6^{\frac{\theta}{2(-1+\theta)}}B\Big(\frac{\sqrt{Q}}{6}\Big)^{\frac{-2+\theta}{-1+\theta}}\theta$$

$$\Big(1-\Big(1+6^{\frac{\theta}{2-2\theta}}B\Big(\frac{\sqrt{Q}}{6}\Big)^{\frac{\theta}{-1+\theta}})\theta\theta+2^{\frac{2+\theta^2}{2}}3^{\frac{1+\theta^2}{2}}BQ^{\frac{3\theta}{2}}\beta nc^2\Big(B\theta\Big)^{\frac{1}{1-\theta}}\kappa^2$$

$$-6^{\frac{3-3\theta+\theta^2}{2}}B\sqrt{Q}\beta nc^2\kappa^2 lnQ\Big]\beta\Big(-2+\theta\Big)\Big(-1+\theta\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\kappa^2-\sqrt{6}Bnc^2\kappa^2 lnQ\Big)$$

$$+2^{\frac{-(-2+\theta)(-1+\theta)}{2}}3^{\frac{-(-3+\theta)\theta}{2}}6^{1-\frac{3\theta}{2}}Q^{1-\frac{3\theta}{2}}\Big(6^{1+\frac{\theta^2}{2}}c_5Q^{\frac{3\theta}{2}}+\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\Big(-6^{\frac{3\theta}{2}}c^2Q^{\frac{1+\theta^2}{2}}$$

$$\beta\Big(-3+\theta\Big)\Big(-1+\theta\Big)^2\theta+2^{\frac{3+\theta^2}{2}}3^{\frac{1+\theta^2}{2}}BQ^{\frac{3\theta}{2}}\beta nc^2\Big(B\theta\Big)^{\frac{1}{\theta-1}}\Big)\kappa^2\Big)-6^{\frac{3-3\theta+\theta^2}{2}}BQ\beta nc^2\kappa^2 lnQ)\Big]\Big[3\sqrt{Q}\Big(12+\frac{6}{\sqrt{}}$$

$$-12c^2\Big(\alpha-\beta\Big)\kappa^2-\frac{2\sqrt{6}B\beta nc^2\kappa^2}{\sqrt{Q}}+6^{\frac{-(-3+\theta)}{2}}c^2Q^{\frac{-(-3+\theta)}{2}}\beta\Big(-2+\theta\Big)\Big(-1+\theta\Big)^3$$

$$\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\kappa^2-\frac{\sqrt{6}B\beta nc^2\kappa^2 lnQ}{\sqrt{Q}}\Big]^{-1}\geq 0 \tag{3.79}$$

For DEC:

$$\Big[Q\Big(6^{1-\frac{3\theta}{2}}\Big)\Big(6^{1+\frac{\theta^2}{2}}\Big)c_5\sqrt{Q}+Q^{\frac{1}{2}-\frac{3\theta}{2}}\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\Big(-6^{\frac{3\theta}{2}}c^2Q^{\frac{1+\theta^2}{2}}\beta\Big(-3+\theta\Big)\Big(-1+\theta\Big)^2\theta$$

$$+2^{\frac{3+\theta^2}{2}}3^{\frac{1+\theta^2}{2}}BQ^{\frac{3\theta}{2}}\beta nc^2\Big(B\theta\Big)^{\frac{1}{1-\theta}}\Big)\Big)\kappa^2-6^{\frac{3-3\theta+\theta^2}{2}}B\sqrt{Q}\beta nc^2\kappa^2 lnQ\Big)\Big]$$

$$\Big[6^{1-\frac{3\theta}{2}}\Big(6^{1+\frac{\theta^2}{2}}c_5\sqrt{Q}-2^{\frac{3+\theta^2}{2}}3^{\frac{1+\theta^2}{2}}B\sqrt{Q}\beta nc^2\kappa^2+6^{\frac{3\theta}{2}}c^2Q^{\frac{1}{2}(-2+\theta)(-1+\theta)}-\beta\Big(-2+\theta\Big)$$

$$\Big(-1+\theta\Big)^3\Big(B\theta\Big)^{\frac{1}{\theta-1}}\kappa^2-2^{2+\frac{\theta^2}{2}}3^{1+\frac{\theta^2}{2}}Q\Big(-1+c^2\Big(\alpha-\beta\Big)\kappa^2\Big)\Big)$$
$$-6^{\frac{3-3\theta+\theta^2}{2}}B\sqrt{Q}\beta nc^2\kappa^2 lnQ\Big]^{-1}\geq 0 \tag{3.80}$$

And,

$$\frac{1}{3}\exp^{2Re[\frac{ln[\frac{\sqrt{Q}}{B\theta}]}{-1+\theta}]}|2Q\Big(\frac{\sqrt{Q}}{6}\Big)+B\Big(\frac{\sqrt{Q}}{B\theta}\Big)^{\frac{\theta}{-1+\theta}}\theta\Big[-6^{\frac{\theta}{2(-1+\theta)}}\Big[+6^{\frac{\theta}{2(-1+\theta)}}+B\Big(\frac{\sqrt{Q}}{B\theta}\Big)^{\frac{\theta}{-1+\theta}}\theta|$$

$$-\Big[Q\Big(6^{1-\frac{3\theta}{2}}\Big(6^{1+\frac{\theta^2}{2}}c_5\sqrt{Q}+Q^{\frac{1}{2}-\frac{3\theta}{2}}\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\Big(-6^{\frac{3\theta}{2}}c^2Q^{\frac{1+\theta^2}{2}}\beta\Big(-3+\theta\Big)\Big(-1+\theta\Big)^2\theta$$

$$+2^{\frac{3+\theta^2}{2}}3^{\frac{1+\theta^2}{2}}BQ^{\frac{3\theta}{2}}\beta nc^2\Big(B\theta\Big)^{\frac{1}{1-\theta}}\Big)\kappa^2\Big)-6^{\frac{3-3\theta+\theta^2}{2}}B\sqrt{Q}\beta nc^2\kappa^2 lnQ\Big)\Big]\Big[6^{1-\frac{3\theta}{2}}\big(6^{1+\frac{\theta^2}{2}}c_5\sqrt{Q}$$

$$-2^{\frac{3+\theta^2}{2}}3^{\frac{1+\theta^2}{2}}BQ^{\frac{3\theta}{2}}\beta nc^2\Big(B\theta\Big)^{\frac{1}{1-\theta}})\kappa^2\Big)-6^{\frac{3-3\theta+\theta^2}{2}}B\sqrt{Q}\beta nc^2\kappa^2 lnQ\Big)\Big]\Big[6^{1-\frac{3\theta}{2}}\Big(6^{1+\frac{\theta^2}{2}}c_5\sqrt{Q}$$

$$-2^{\frac{3+\theta^2}{2}}3^{\frac{1+\theta^2}{2}}B\sqrt{Q}\beta nc^2\kappa^2+6^{\frac{3\theta}{2}}c^2Q^{\frac{(-2+\theta)(-1+\theta)}{2}}\beta\Big(-2+\theta\Big)\Big(-1+\theta\Big)^3\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\kappa^2$$
$$-2^{2+\frac{\theta^2}{2}}3^{1+\frac{\theta^2}{2}}Q\Big(-1+c^2\Big(\alpha-\beta\Big)\kappa^2\Big)\Big]-6^{\frac{3-3\theta+\theta^2}{2}}B\sqrt{Q}\beta nc^2\kappa^2 lnQ\Big]^{-1}\leq 0 \tag{3.81}$$

For SEC:

$$\Big[2Q\Big(6c_5+12\sqrt{Q}-12c^2\sqrt{Q}\Big(\alpha-\beta\Big)\kappa^2-2\sqrt{6}\beta nc^2\kappa^2+6^{\frac{-(-3+\theta)\theta}{2}}c^2Q^{\frac{1-3\theta+\theta^2}{2}}\beta\Big(-2+\theta\Big)$$

$$\Big(-1+\theta\Big)^3\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\kappa^2-\sqrt{6}B\beta nc^2\kappa^2 lnQ\Big)-6^{\frac{\theta}{2(-1+\theta)}}B\Big(\frac{\sqrt{Q}}{B\theta}\Big)^{\frac{-2+\theta}{-1+\theta}}\theta\Big[1-\Big(1+6^{\frac{\theta}{2-2\theta}}$$

$$B\Big(\frac{\sqrt{Q}}{B\theta}\Big)^{\frac{\theta}{-1+\theta}}\Big)\theta\Big]\Big[6c_5+12\sqrt{Q}-12c^2\sqrt{Q}\Big(\alpha-\beta\Big)\kappa^2-2\sqrt{6}B\beta nc^2\kappa^2+6^{\frac{-(-3+\theta)\theta}{2}}$$

$$c^2Q^{\frac{1-3\theta+\theta^2}{2}}\beta\Big(-2+\theta\Big)\Big(-1+\theta\Big)^3\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\kappa^2-\sqrt{6}B\beta nc^2\kappa^l nQ\Big]+6^{\frac{-(-2+\theta)(-1+\theta)}{2}}$$

$$\Big[6^{1-\frac{3\theta}{2}}Q^{1-\frac{3\theta}{2}}\Big(6^{1+\frac{\theta^2}{2}}c_5Q^{\frac{3\theta}{2}}+\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\Big(-6^{\frac{3\theta}{2}}c^2Q^{\frac{1+\theta^2}{2}}\beta\Big(-3+\theta\Big)\Big(-1+\theta\Big)^2\theta$$

$$+2^{\frac{3+\theta^2}{2}}3^{\frac{1+\theta^2}{2}}BQ^{\frac{3\theta}{2}}\beta nc^2\Big(B\theta\Big)^{\frac{1}{1-\theta}}\Big)\kappa^2\Big)-6^{\frac{3-3\theta+\theta^2}{2}}BQ\beta nc^2\kappa^2 lnQ\Big)\Big]\Big[6c_5+12\sqrt{Q}$$

$$-12c^2\sqrt{Q}\Big(\alpha-\beta\Big)\kappa^2-2\sqrt{6}B\beta nc^2\kappa^2+6^{\frac{-(-3+\theta)\theta}{2}}c^2Q^{\frac{1-3\theta+\theta^2}{2}}\beta\Big(-2+\theta\Big)\Big(-1+\theta\Big)^3$$
$$\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\kappa^2-\sqrt{6}B\beta nc^2\kappa^2 lnQ\Big]^{-1}\geq 0 \tag{3.82}$$

3.5.4 Case IV :

Considering the scale factor $a(t) = A\Big(B + e^{nt}\Big)^{\lambda}$.

Here we will solve the energy condition and we will get the inequalities WEC, NEC, DEC,SEC, for the GO models:

$$f(Q) = c_7\sqrt{Q} + \Big[1 - \kappa^2 c^2\Big(\alpha - \frac{\beta}{\lambda}\Big)Q - c^2\kappa^2\beta n\Big]\sqrt{6Q}$$

$$f'(Q) = \frac{c_7}{2\sqrt{Q}} + \Big(1 - c^2\kappa^2 - \alpha - \frac{\beta}{\lambda}\Big) - \frac{3c^2\kappa^2\beta n}{\sqrt{6}Q}$$

Now,

$$f'(Q) = f''(Q)\dot{Q}$$

$$= 12\Big(\frac{\sqrt{Q}}{6}\Big)\Big(\frac{BQ}{e^{nt}6\lambda^2 n^2}\Big)$$

Again,

$$\rho_{de} = \frac{3}{\kappa^2}\Big[\frac{Q}{6}\Big(1 - 2f_Q\Big) + \frac{f}{6}\Big)\Big]$$

$$\rho_{de} = \frac{1}{2\kappa^2}\Big[Q(1 - 2f_Q) + f\Big]$$

$$= \frac{1}{2\kappa^2}\Big[Q\Big(1 - \frac{c_7}{\sqrt{Q}} - 2 + 2 - c^2\kappa^2\alpha - \frac{\beta}{n}\Big) + c_7\sqrt{Q} + \Big(1 - c^2\kappa^2\Big(\alpha - \frac{\beta}{\lambda}\Big)Q\Big) - c^2\kappa^2\beta n\Big]$$

$$= \frac{1}{2\kappa^2}\Big[\Big(-1 - \frac{c_7}{\sqrt{Q}} - c^2\kappa^2\Big(\alpha - \frac{\beta}{\lambda}\Big)Q + c_7\sqrt{Q} + \Big(1 - c^2\kappa^2\Big(\alpha - \frac{\beta}{\lambda}\Big)Q\Big) - c^2\kappa^2\beta n\sqrt{\frac{Q}{6}}\Big]$$

$$= \frac{1}{2\kappa^2} - c^2\kappa^2\beta n\sqrt{\frac{Q}{6}}$$

$$\rho_{de} = \frac{c^2}{2}\beta n\Big(\frac{Q}{6}\Big) \tag{3.83}$$

$$\rho_{de} = -\frac{1}{\kappa^2}\Big[2\dot{H}\Big(1 - 2f_Q\Big) + \frac{f}{2} + 3H^2\Big(1 - 8f_{QQ}\dot{H} - 2f_Q\Big)\Big]$$

$$= -\frac{1}{\kappa^2}\Big[2\Big(\frac{BQ}{e^{nt}6\lambda^2n^2}\Big)\Big[\Big(-1-\frac{c_7}{\sqrt{Q}}-c^2\kappa^2\Big(\alpha-\frac{\beta}{\lambda}\Big)Q+c_7\sqrt{Q}+\Big(1-c^2\kappa^2\Big(\alpha-\frac{\beta}{\lambda}\Big)Q\Big)-$$

$$c^2\kappa^2\beta n\sqrt{\frac{Q}{6}}+3\Big(\frac{Q}{6}\Big)\Big(1-8\Big(\frac{c_7}{2\sqrt{Q}}\Big)+\Big(1-c^2\kappa^2\Big(\alpha-\frac{\beta}{\lambda}\Big)Q\Big)-c^2\kappa^2\beta n\frac{-3c^2\kappa^2\beta n}{6\sqrt{Q}}$$

$$+2\Big(-2c^2\kappa^2\Big(\alpha-\frac{\beta}{\lambda}\Big)-c^2\kappa^2\beta n\sqrt{6Q}\Big]$$

$$= -\frac{1}{\kappa^2}\Big[\Big(\frac{2BQ}{e^{nt}3\lambda^2n^2}\Big)\Big(-1-\frac{c_7}{\sqrt{Q}}-c^2\kappa^2\Big(\alpha-\frac{\beta}{\lambda}\Big)Q+c_7\sqrt{Q}+\Big(1-c^2\kappa^2\Big(\alpha-\frac{\beta}{\lambda}\Big)Q\Big)-$$

$$c^2\kappa^2\beta n\sqrt{\frac{Q}{6}}+3\Big(\frac{Q}{6}\Big)\Big(1-8\Big(\frac{c_7}{2\sqrt{Q}}\Big)+\Big(1-c^2\kappa^2\Big(\alpha-\frac{\beta}{\lambda}\Big)Q\Big)-\frac{2c_7}{\sqrt{Q}}\frac{c^2}{2}\beta n\Big(\frac{Q}{6}\Big)$$

$$= \frac{-c^2}{n^2}\Big(3n-2\Big)\Big(\alpha-\frac{\beta}{\lambda}\Big)Q-c^2\kappa^2\beta n\sqrt{6Q} \tag{3.84}$$

Again,

$$\rho_{eff} = \frac{1}{f_Q}\Big(\rho+\frac{f}{2}\Big)$$

$$= \frac{1}{f_Q}\Big(\rho_{de}+\frac{f}{2}\Big)$$

$$= \Big[c_7\sqrt{Q}+\Big(1-c^2\kappa^2\Big(\alpha-\frac{\beta}{\lambda}\Big)\Big)Q-c^2\kappa^2\beta n\sqrt{Q}+\frac{c^2\sqrt{Q}}{2}\beta n\Big(\frac{Q}{6}\Big)\frac{Q}{2}\Big]$$

$$\Big[1+\frac{c_7}{2\sqrt{Q}}+\frac{3c^2\kappa^2n\beta}{2\sqrt{Q}}-c^2\kappa^2\Big(\alpha-\frac{\beta}{\lambda}\Big)+\frac{\sqrt{3}c^2\kappa^2n\beta}{2\sqrt{2}Q}logQ\Big]^{-1} \tag{3.85}$$

Now,

$$p_{eff} = -2\frac{f(\dot{Q})}{f'(Q)}H+\frac{1}{f'(Q)}\Big(p+\frac{f}{2}\Big)$$

$$= -2\frac{f''(Q)}{f'(Q)}\dot{Q}H+\frac{1}{f'(Q)}\Big(p_{de}+\frac{f}{2}\Big)$$

$$= -2\frac{f''(Q)}{f'(Q)}(12H\dot{H})H+\frac{1}{f'(Q)}\Big(p_{de}+\frac{f}{2}\Big)$$

$$= -24\frac{f''(Q)}{f'(Q)}H^2\dot{H} + \frac{1}{f'(Q)}\Big(p_{de} + \frac{f}{2}\Big)$$

$$= -24\frac{f''(Q)}{f'(Q)}\Big(\frac{Q}{6}\Big)\Big(\frac{-Q}{6n}\Big) + \frac{1}{f'(Q)}\Big(p_{de} + \frac{f}{2}\Big)$$

$$p_{eff} = \frac{2Q^2}{3n}\frac{f''(Q)}{f'(Q)} + \frac{1}{f'(Q)}\Big(\rho_{de} + \frac{f}{2}\Big)$$

$$= \frac{-c_7\sqrt{Q}}{2\sqrt{Q}} - \frac{c^2}{n^2}\Big(3n-2\Big)\Big(\alpha - \frac{\beta}{\lambda}\Big)Q - c^2\kappa^2 n\beta\sqrt{6}Q + \frac{1}{2}\Big[1 - c^2\kappa^2\Big(\alpha - \frac{\beta}{\lambda}\Big)$$

$$-c^2\kappa^2 n\beta\sqrt{6Q}\Big]\Big[1 + \frac{c_7}{2\sqrt{Q}} + \frac{3c^2\kappa^2\beta n}{\sqrt{2}Q} - c^2\kappa^2\Big(\alpha - \frac{\beta}{\lambda}\Big) + \frac{\sqrt{3}c^2\kappa^2\beta n}{2\sqrt{2}Q}logQ\Big]^{-1} \quad (3.86)$$

For WEC:

$$\Big[c_7\sqrt{Q} + Q\Big(1 - c^2\kappa^2\Big(\alpha - \frac{\beta}{\lambda}\Big)\Big) + \sqrt{\frac{3}{2}}c^2 n\sqrt{Q}\beta\kappa^2 - Q\Big(1 + \frac{c_7}{2\sqrt{Q}} + \frac{\sqrt{3}c^2\kappa^2 n\beta}{\sqrt{2Q}}$$

$$-c^2\kappa^2\Big(\alpha - \frac{\beta}{\lambda}\Big) + \frac{\sqrt{3}c^2\kappa^2 n\beta logQ}{2\sqrt{Q}}\Big)\Big]\Big[1 + \frac{c_7}{2\sqrt{Q}} + \frac{\sqrt{3}c^2\kappa^2 n\beta}{2\sqrt{Q}} - c^2\kappa^2\Big(\alpha - \frac{\beta}{\lambda}\Big)$$

$$+\frac{\sqrt{3}c^2\kappa^2 n\beta logQ}{2\sqrt{2Q}}^{-1} \geq 0 \quad (3.87)$$

$$\Big[\Big(2\Big(\lambda\Big(2\sqrt{Q}\Big(3\lambda - 1 + \sqrt{\frac{6}{Q}}n\lambda\Big) + c_7\Big(6\lambda - 1 + \frac{\sqrt{6}}{Q}n\lambda\Big)\Big) + c^2\kappa^2\Big(n\beta\lambda\Big(\frac{6n\lambda}{\sqrt{Q}} - \sqrt{Q}\Big)$$

$$+2\sqrt{Q}\Big(3\lambda - 1 + \sqrt{\frac{6}{Q}}n\lambda\Big)\Big(\beta - \alpha\lambda\Big)\Big)\Big) + c^2 n\beta\kappa^2\lambda\Big(-\sqrt{6}\Big(\sqrt{6} + \frac{n}{\sqrt{Q}}\Big)\lambda\Big)logQ\Big)\Big]$$

$$\Big[3\lambda\Big(2\Big(\Big(c_7 + 2\sqrt{Q}\Big)\lambda + c^2\kappa^2\Big(\sqrt{6}n\beta\lambda + 2\sqrt{Q}\Big(\beta - \alpha\lambda\Big)\Big)\Big) + \sqrt{6}c^2\kappa^2 n\beta\lambda logQ\Big)\Big]^{-1} \geq 0 \quad (3.88)$$

For NEC:

$$\Big[Q\Big(2\Big(\lambda\Big(2\sqrt{Q}\Big(3\lambda - 1 + \sqrt{\frac{6}{Q}}n\lambda\Big)\Big) + c^2\kappa^2\Big(n\beta\lambda\Big(\frac{6n\lambda}{\sqrt{Q}} - \sqrt{6}\Big) + 2\sqrt{Q}\Big(3\lambda - 1Q$$

$$+\sqrt{\frac{6}{Q}}n\lambda\Big)\Big(\alpha - \beta\Big)\Big)\Big) + c^2\kappa^2 n\beta\lambda\Big(-\sqrt{6} + 6\Big(\sqrt{6} + \frac{n}{Q}\Big)\lambda\Big)logQ\Big)\Big]\Big[3\lambda\Big(2\Big(\Big(c_7 + 2\sqrt{Q}\Big)$$

$$\lambda + c^2\kappa^2\Big(\sqrt{6}n\beta\lambda + 2\sqrt{Q}\Big(\beta - \alpha\lambda\Big)\Big)\Big) + \sqrt{6}c^2\kappa^n\beta\lambda logQ\Big)\Big]^{-1} \geq 0 \tag{3.89}$$

For DEC:

$$\Big[c_7\sqrt{Q} + Q\Big(1 - c^2\kappa^2\Big(\alpha - \frac{\beta}{\lambda}\Big)\Big) + \sqrt{\frac{3}{2}}c^2n\sqrt{Q}\beta\kappa^2 logQ - Q\Big(1 + \frac{c_7}{2\sqrt{Q}} + \frac{\sqrt{3}c^2\kappa^2n\beta}{\sqrt{2Q}}$$

$$-c^2\kappa^2\Big(\alpha - \frac{\beta}{\lambda}\Big) + \frac{\sqrt{3}c^2\kappa^2n\beta logQ}{2\sqrt{Q}}\Big)\Big]\Big[1 + \frac{c_7}{2\sqrt{Q}} + \frac{\sqrt{3}c^2\kappa^2n\beta}{\sqrt{2Q}} - c^2\kappa^2\Big(\alpha - \frac{\beta}{\lambda}\Big)$$

$$+\frac{\sqrt{3}c^2\kappa^2n\beta logQ}{2\sqrt{Q}}\Big] \geq 0$$

$$\Big|\sqrt{2Q}3n + Q - \frac{Q}{3\lambda}\Big| -$$

$$\frac{Q\lambda\Big(2c_7 - 2\sqrt{6}c^2\beta n\kappa^2 + \sqrt{6}c^2\kappa^2n\beta logQ\Big)}{2\Big[\lambda\Big(c_7 + 2\sqrt{Q}\Big)c^2\kappa^2\Big(\sqrt{6}n\beta\lambda + 2\sqrt{Q}\Big(\beta - \alpha\lambda\Big)\Big)\Big] + \sqrt{6}c^2\kappa^2n\beta\lambda logQ}\Big]^{-1} \leq 0 \tag{3.90}$$

For SEC:

$$\Big[Q\Big(2\Big(\lambda\Big(2\sqrt{Q}\Big(3\lambda - 1 + \sqrt{\frac{6}{Q}}n\lambda\Big) + c_7\Big(4\lambda - 1 + \sqrt{\frac{6}{Q}}n\lambda\Big)\Big) + c^2\kappa^2\Big(n\beta\lambda\Big(\frac{6n\lambda}{\sqrt{Q}}$$

$$-\sqrt{6} + 2\sqrt{6}\lambda\Big) + 2\sqrt{Q}\Big(3\lambda - 1 + \sqrt{\frac{6}{Q}}n\lambda\Big)\Big(\beta - \alpha\lambda\Big)\Big)\Big) + c^2\kappa^2n\beta\lambda\Big(-\sqrt{6} + 4\sqrt{6}\lambda$$

$$+\frac{6n}{\sqrt{Q}}\lambda\Big)logQ\Big)\Big]\Big[\lambda\Big(2\Big(\Big(c_7 + 2\sqrt{Q}\Big)\lambda + c^2\kappa^2\Big(\sqrt{6}n\beta\lambda + 2\sqrt{Q}\Big(\beta - \alpha\lambda\Big)\Big)\Big)$$

$$+\sqrt{6}c^2\kappa^2n\beta\lambda logQ\Big)\Big]^{-1} \geq 0 \tag{3.91}$$

- **Chen-Jing Model:**

3.5.5 Case I:

Here we will obtained the expression using Chen-Jing Model equation (3.26).

$$f(Q) = c_2\sqrt{Q} + \Big[1 - \frac{\kappa^2c^2}{m^2}\Big(2\alpha - m\beta + m^2\gamma\Big)\Big]Q$$

$$f'(Q) = \frac{c_2}{2\sqrt{Q}} + 1 - \frac{c^2\kappa^2}{m}\Big(2\alpha - m\beta + m^2\gamma\Big)$$

Now,

$$f'(Q) = f''(Q)\dot{Q} = \frac{1}{12mQ}$$

Again,

$$\rho_{de} = \frac{3}{\kappa^2}\Big[\frac{Q}{6}\Big(1 - 2f_a\Big) + \frac{f}{6}\Big]$$

$$\rho_{de} = \frac{1}{2\kappa^2}\Big[Q(1 - 2f_a) + f\Big]$$

$$= \frac{1}{2\kappa^2}\Big[Q\Big(1 - 2c_2\sqrt{Q} + 1 - -\frac{c^2\kappa^2}{m}\Big(2\alpha - m\beta + m^2\gamma\Big) + \frac{1}{2\sqrt{Q}} + 1$$

$$-\frac{c^2\kappa^2}{m}\Big(2\alpha - m\beta + m^2\gamma\Big)\Big]$$

$$= \frac{1}{2\kappa^2}\Big[Q\Big(1 - 2c_2\sqrt{Q} + 2 - \frac{c^2\kappa^2}{m^2} - 4\alpha - 2m\beta + 2m^2\gamma\Big) + \frac{1}{2\sqrt{Q}} + 1$$

$$-\frac{c^2\kappa^2}{m}\Big(2\alpha - m\beta + m^2\gamma\Big)\Big]$$

$$= \frac{1}{2\kappa^2}\frac{c^2\kappa^2}{m}\Big(2\alpha - m\beta + m^2\gamma\Big)Q$$

$$\rho_{de} = \frac{c^2}{2m}\Big(2\alpha - m\beta + m^2\gamma\Big)Q \tag{3.92}$$

Again,

$$\rho_{de} = -\frac{1}{\kappa^2}\Big[2\dot{H}\Big(1 - 2f_Q\Big) + \frac{f}{2} + 3H^2\Big(1 - 8f_{QQ}\dot{H} - 2f_Q\Big)\Big]$$

$$= \frac{1}{2\kappa^2}\Big[\Big(\frac{-Q}{3m}\Big) + \frac{1}{2\kappa^2}\Big[Q\Big(1 - 2c_2\sqrt{Q} + 1 - -\frac{c^2\kappa^2}{m}\Big(2\alpha - m\beta + m^2\gamma\Big) + \frac{1}{2\sqrt{Q}} + 1$$

$$-\frac{c^2\kappa^2}{m}\Big(2\alpha - m\beta + m^2\gamma\Big)\Big] + \frac{c_2\sqrt{Q}}{2} + \frac{3Q}{2} + 8\Big(1 - 2c_2\sqrt{Q} + 2 - \frac{c^2\kappa^2}{m^2} - 4\alpha - 2m\beta$$

$$+2m^2\gamma\Big) + \frac{1}{2\sqrt{Q}} + \frac{1}{2\kappa^2}\Big[Q\Big(1 - 2c_2\sqrt{Q} + 2 - \frac{c^2\kappa^2}{m^2} - 4\alpha - 2m\beta + 2m^2\gamma\Big) + \frac{1}{2\sqrt{Q}} +$$

$$\frac{c^2\kappa^2}{m}\Big(2\alpha - m\beta + m^2\gamma\Big)$$

$$= \frac{-1}{\kappa^2}\Big[\ \frac{c^2\kappa^2}{m^2}\Big(2\alpha - m\beta + m^2\gamma\Big)Q\Big(\frac{m}{2} - \frac{1}{3}\Big)\Big]$$

$$\rho_{de} = \frac{c^2}{6m^2}\Big(3m - 2\Big)\Big(2\alpha - m\beta + m^2\gamma\Big)Q \tag{3.93}$$

Now,

$$\rho_{eff} = \frac{1}{f_Q}\Big(\rho + \frac{f}{2}\Big)$$

$$= \frac{1}{f_Q}\Big(\rho_{de} + \frac{f}{2}\Big)$$

$$= \frac{c^2}{2m}\Big(2\alpha - m\beta + m^2\gamma\Big)Q + \frac{1}{2\sqrt{Q}} + 1\frac{c^2\kappa^2}{m^2}\Big(2\alpha - m\beta + m^2\gamma\Big) +$$

$$\frac{Q}{2}\Big[\frac{1}{2\sqrt{Q}} + 1 - \frac{c^2\kappa^2}{m}\Big(2\alpha - m\beta + m^2\gamma\Big)\Big]^{-1} \tag{3.94}$$

Again,

$$p_{eff} = -2\frac{f(\dot{Q})}{f'(Q)}H + \frac{1}{f'(Q)}\Big(p + \frac{f}{2}\Big)$$

$$= -2\frac{f''(Q)}{f'(Q)}\dot{Q}H + \frac{1}{f'(Q)}\Big(p_{de} + \frac{f}{2}\Big)$$

$$= -2\frac{f''(Q)}{f'(Q)}(12H\dot{H})H + \frac{1}{f'(Q)}\Big(p_{de} + \frac{f}{2}\Big)$$

$$= -24\frac{f''(Q)}{f'(Q)}H^2\dot{H} + \frac{1}{f'(Q)}\Big(p_{de} + \frac{f}{2}\Big)$$

$$= -24\frac{f''(Q)}{f'(Q)}\Big(\frac{Q}{6}\Big)\Big(\frac{-Q}{6m}\Big) + \frac{1}{f'(Q)}\Big(p_{de} + \frac{f}{2}\Big)$$

$$p_{eff} = \frac{2Q^2}{3m}\frac{f''(Q)}{f'(Q)} + \frac{1}{f'(Q)}\Big(\rho_{de} + \frac{f}{2}\Big)$$

$$= \frac{2Q}{3m}\frac{1}{12mQ} - \frac{c^2}{6m}\Big(3m-2\Big)\Big(2\alpha - m\beta + m^2\gamma\Big) + \frac{Q}{2}\Big[\frac{1}{2\sqrt{Q}} + 1 - \frac{c^2\kappa^2}{m}\Big(2\alpha - m\beta + m^2\gamma\Big)\Big]^{-1} \tag{3.95}$$

For WEC:

$$\frac{c_2\Big(\sqrt{Q} - \frac{1}{2\sqrt{Q}}\Big)}{1 + \frac{c_2}{2\sqrt{Q}} - \frac{c^2\kappa^2(2\alpha + m\beta + m^2\gamma)}{m^2}} \geq 0 \tag{3.96}$$

$$-\frac{Q\Big[c_2\sqrt{Q}(m^2 - 6m^3) + 2(3m-1)\sqrt{Q} + 2c^2\kappa^2\alpha - c^2\kappa^2 m\beta + m^2(c^2\gamma\kappa^2 - 1)\Big]}{3m\Big[c_2 m^2 + 2\sqrt{Q} - 2c^2\kappa^2\alpha + c^2\kappa^2 m\beta + m^2(1 - c^2\kappa^2\gamma)\Big]} \geq 0 \tag{3.97}$$

For NEC:

$$-\frac{Q\Big[c_2\sqrt{Q}(m^2 - 6m^3) + 2(3m-1)\sqrt{Q} + 2c^2\kappa^2\alpha - c^2\kappa^2 m\beta + m^2(c^2\gamma\kappa^2 - 1)\Big]}{3m\Big[c_2 m^2 + 2\sqrt{Q} - 2c^2\kappa^2\alpha + c^2\kappa^2 m\beta + m^2(1 - c^2\kappa^2\gamma)\Big]} \geq 0 \tag{3.98}$$

For DEC:

$$\frac{c_2\Big(\sqrt{Q} - \frac{1}{2\sqrt{Q}}\Big)}{1 + \frac{c_2}{2\sqrt{Q}} - \frac{c^2\kappa^2(2\alpha + m\beta + m^2\gamma)}{m^2}} \geq 0 \tag{3.99}$$

$$-\frac{c_2\sqrt{Q}}{2\Big(1 + \frac{c_2}{2\sqrt{Q}} - \frac{c^2(2\alpha - m\beta + m^2\gamma)\kappa^2}{m^2}\Big)} + |Q - \frac{Q}{3m}| \leq 0 \tag{3.100}$$

For SEC:

$$-\frac{Q\Big[c_2(m - 4m^2) + 2(-1+3m)\sqrt{Q}(2c^2\kappa^2\alpha - c^2\kappa^2 m\beta) + m^2(-1 + c^2\kappa^2\gamma)\Big]}{m\Big[c_2 m^2 + 2\sqrt{Q}(-2c^2\kappa^2\alpha + c^2 m\beta\kappa^2) + m^2(1 - c^2\kappa^2\gamma)\Big]} \tag{3.101}$$

3.5.6 Case II:

Here we will obtained the expression using Chen-Jing Model equation [3.35]:

$$f(Q) = c_4\sqrt{Q} + \Big[1 - \frac{c^2\kappa^2}{n^2}\Big(2\alpha + \beta n + \gamma n^2\Big)\Big]Q$$

$$f'(Q) = f'(Q) = \frac{c_4}{2\sqrt{Q}} + 1 - \frac{c^2\kappa^2}{m}\Big(2\alpha + n\beta + n^2\gamma\Big)$$

Now,

$$f'(Q) = f''(Q)\dot{Q} = \frac{1}{12nQ}$$

Again,

$$\rho_{de} = \frac{3}{\kappa^2}\Big[\frac{Q}{6}\Big(1-2f_a\Big)+\frac{f}{6}\Big]$$

$$\rho_{de} = \frac{1}{2\kappa^2}\Big[Q(1-2f_a)+f\Big]$$

$$= \frac{1}{2\kappa^2}\Big[Q\Big(1-2c_4\sqrt{Q}+1-\frac{c^2\kappa^2}{n}\Big(2\alpha+n\beta+n^2\gamma\Big)+\frac{1}{2\sqrt{Q}}+1$$

$$-\frac{c^2\kappa^2}{n}\Big(2\alpha+n\beta+n^2\gamma\Big)\Big]$$

$$= \frac{1}{2\kappa^2}\Big[Q\Big(1-2c_4\sqrt{Q}+2-\frac{c^2\kappa^2}{n^2}-4\alpha+2n\beta+2n^2\gamma\Big)+\frac{1}{2\sqrt{Q}}+1$$

$$-\frac{c^2\kappa^2}{n}\Big(2\alpha+n\beta+n^2\gamma\Big)\Big]$$

$$= \frac{1}{2\kappa^2}\frac{c^2\kappa^2}{n}\Big(2\alpha+n\beta+n^2\gamma\Big)Q$$

$$\rho_{de} = \frac{c^2}{2n}\Big(2\alpha+n\beta+n^2\gamma\Big)Q \tag{3.102}$$

Again,

$$\rho_{de} = -\frac{1}{\kappa^2}\Big[2\dot{H}\Big(1-2f_Q\Big)+\frac{f}{2}+3H^2\Big(1-8f_{QQ}\dot{H}-2f_Q\Big)\Big]$$

$$= \frac{1}{2\kappa^2}\Big[\Big(\frac{-Q}{3n}\Big)+\frac{1}{2\kappa^2}\Big[Q\Big(1-2c_4\sqrt{Q}+1-\frac{c^2\kappa^2}{n}\Big(2\alpha+n\beta+n^2\gamma\Big)+\frac{1}{2\sqrt{Q}}+1$$

$$-\frac{c^2\kappa^2}{n}\Big(2\alpha+n\beta+n^2\gamma\Big)\Big]+\frac{c_4\sqrt{Q}}{2}+\frac{3Q}{2}+8\Big(1-2c_4\sqrt{Q}+2-\frac{c^2\kappa^2}{n^2}-4\alpha+2n\beta$$

$$+2n^2\gamma\Big)+\frac{1}{2\sqrt{Q}}+\frac{1}{2\kappa^2}\Big[Q\Big(1-2c_4\sqrt{Q}+2-\frac{c^2\kappa^2}{m^2}-4\alpha-2n\beta+2n^2\gamma\Big)+\frac{1}{2\sqrt{Q}}+$$

$$\frac{c^2\kappa^2}{n}\Big(2\alpha+n\beta+n^2\gamma\Big)$$

$$= \frac{-1}{\kappa^2}\Big[\ \frac{c^2\kappa^2}{n^2}\Big(2\alpha + n\beta + n^2\gamma\Big)Q\Big(\frac{n}{2} - \frac{1}{3}\Big)\Big]$$

$$\rho_{de} = \frac{c^2}{6n^2}\Big(3n-2\Big)\Big(2\alpha + n\beta + n^2\gamma\Big)Q \tag{3.103}$$

Now,

$$\rho_{eff} = \frac{1}{f_Q}\Big(\rho + \frac{f}{2}\Big)$$

$$= \frac{1}{f_Q}\Big(\rho_{de} + \frac{f}{2}\Big)$$

$$= \frac{c^2}{2n}\Big(2\alpha + n\beta + n^2\gamma\Big)Q + \frac{1}{2\sqrt{Q}} + \frac{c^2\kappa^2}{n^2}\Big(2\alpha + n\beta + n^2\gamma\Big) + \frac{Q}{2}\Big[\frac{1}{2\sqrt{Q}} + 1 - \frac{c^2\kappa^2}{n}\Big(2\alpha + n\beta + n^2\gamma\Big)\Big]^{-1} \tag{3.104}$$

Again,

$$p_{eff} = -2\frac{f(\dot{Q})}{f'(Q)}H + \frac{1}{f'(Q)}\Big(p + \frac{f}{2}\Big)$$

$$= -2\frac{f''(Q)}{f'(Q)}\dot{Q}H + \frac{1}{f'(Q)}\Big(p_{de} + \frac{f}{2}\Big)$$

$$= -2\frac{f''(Q)}{f'(Q)}(12H\dot{H})H + \frac{1}{f'(Q)}\Big(p_{de} + \frac{f}{2}\Big)$$

$$= -24\frac{f''(Q)}{f'(Q)}H^2\dot{H} + \frac{1}{f'(Q)}\Big(p_{de} + \frac{f}{2}\Big)$$

$$= -24\frac{f''(Q)}{f'(Q)}\Big(\frac{Q}{6}\Big)\Big(\frac{-Q}{6n}\Big) + \frac{1}{f'(Q)}\Big(p_{de} + \frac{f}{2}\Big)$$

$$p_{eff} = \frac{2Q^2}{3n}\frac{f''(Q)}{f'(Q)} + \frac{1}{f'(Q)}\Big(\rho_{de} + \frac{f}{2}\Big)$$

$$= \frac{2Q}{3n}\frac{1}{12nQ} - \frac{c^2}{6n}\Big(3n-2\Big)\Big(2\alpha + n\beta + n^2\gamma\Big) + \frac{Q}{2}\Big[\frac{1}{2\sqrt{Q}} + 1 - \frac{c^2\kappa^2}{n}\Big(2\alpha + n\beta + n^2\gamma\Big)\Big]^{-1} \tag{3.105}$$

For WEC:

$$\frac{c_4 n^2 Q}{c_4 n^2 - 2\sqrt{Q}\Big[2c^2\kappa^2\alpha + c^2\kappa^2 n\beta + n^2\Big(-1 + c^2\kappa^2\gamma\Big)\Big]} \tag{3.106}$$

$$\frac{Q\Big[-c_4 n^2(1+6n) + 2(1+3n)\sqrt{Q} + 2c^2\kappa^2\alpha + c^2\kappa^2 n\beta + n^2(-1+c^2\kappa^2\gamma)\Big]}{3n\Big[-c_4 n^2 + 2\sqrt{Q} + 2c^2\kappa^2\alpha + c^2\kappa^2 n\beta + n^2(-1+c^2\kappa^2\gamma)\Big]} \geq 0 \tag{3.107}$$

For NEC:

$$\frac{Q\Big[-c_4 n^2(1+6n) + 2(1+3n)\sqrt{Q} + 2c^2\kappa^2\alpha + c^2\kappa^2 n\beta + n^2(-1+c^2\kappa^2\gamma)\Big]}{3n\Big[-c_4 n^2 + 2\sqrt{Q} + 2c^2\kappa^2\alpha + c^2\kappa^2 n\beta + n^2(-1+c^2\kappa^2\gamma)\Big]} \geq 0 \tag{3.108}$$

For DEC:

$$\frac{c_4 n^2 Q}{c_4 n^2 - 2\sqrt{Q}\Big[2c^2\kappa^2\alpha + c^2\kappa^2 n\beta + n^2(-1+c^2\kappa^2\gamma)\Big]} \geq 0 \tag{3.109}$$

$$-\frac{c_4\sqrt{Q}}{2\Big[1 + \frac{c_4}{2\sqrt{Q}} - \frac{c^2[2\alpha + n(\beta+n\gamma)]\kappa^2}{n^2}\Big]} + |Q + \frac{Q}{3n}| \leq 0 \tag{3.110}$$

For SEC:

$$\frac{Q\Big[-c_4 n^2(1+4n) + 2(1+3n)\sqrt{Q} + 2c^2\kappa^2\alpha + c^2\kappa^2\beta n + n^2(-1+c^2\kappa^2\gamma)\Big]}{n\Big[-c_4 n^2 + 2\sqrt{Q} + 2c^2\kappa^2\alpha + c^2\kappa^2 n\beta + n^2(-1+c^2\kappa^2\gamma)\Big]} \geq 0 \tag{3.111}$$

3.5.7 Case III:

Here we will cosider the Chen-Jing Model from the equation 3.40.

$$\Rightarrow \quad f(Q) = c_5\sqrt{Q}\Big(1 - c^2\kappa^2\Big)Q - c^2\kappa^2\beta\Big(B\theta\Big)^{\frac{1}{\theta-1}}\Big(\theta - 1\Big)^2\Big(\frac{Q}{6}\Big)^{\frac{\theta-2}{2(\theta-1)}} - c^2\kappa^2\beta B\sqrt{\frac{Q}{6}}lnQ$$

$$f'(Q) = c^5\Big[\frac{1}{2}Q^{\frac{-1}{2}}\Big(1 - c^2\kappa^2\Big)^Q + Q^{\frac{1}{2}}\Big(1 - c^2\kappa^2\Big)^Q ln\Big(1 - c^2\kappa^2\Big) + \frac{-c^2\kappa^2\beta\Big(B\theta\Big)^{\frac{1}{\theta}}\Big(\theta - 1\Big)}{2}$$

$$\Big(\theta - 2\Big)\Big(\frac{1}{6}\Big)^{\theta-2}Q^{\theta-3} - c^2\kappa^2\beta B\frac{1}{\sqrt{6}}\Big(\frac{1}{2}Q^{-\frac{1}{2}}ln(Q) + Q^{-\frac{1}{2}}\Big)$$

Now,

$$f'(Q) = f''(Q)\dot{Q} = \frac{c^2\kappa^2\beta B}{24Q^{\frac{3}{2}}}$$

Again,

$$\rho_{de} = \frac{3}{\kappa^2}\Big[\frac{Q}{6}\Big(1-2f_Q\Big)+\frac{f}{6}\Big)\Big]$$

$$\rho_{de} = \frac{1}{2\kappa^2}\Big[Q(1-2f_Q)+f\Big]$$

$$= \Big[1-c^2\kappa^2\Big(\gamma+2\Big)-6c^2\kappa^2\Big(B\theta\Big)^{\frac{2}{\theta-1}}\Big(\theta-1\Big)\Big(\theta-2\Big)+\alpha^2c^2\kappa^2-6\beta^2c^2\kappa^2\Big(B\theta\Big)\Big(\frac{1}{\theta-1}\Big)$$

$$+\Big(\frac{Q}{6}\Big)^{\frac{\theta-2}{2(\theta-1)}}+\alpha^2c^2\kappa^2\Big(\frac{\theta-1}{3-\theta}\Big)+\frac{\alpha}{3m}-\beta B\theta\Big(\theta-1\Big)+\gamma\sqrt{\frac{Q}{6}}\Big]$$

$$= \frac{1}{2}\Big[1-c^2\kappa^2\Big(\gamma+2\Big)-6c^2\kappa^2\Big(B\theta\Big)^{\frac{2}{\theta-1}}\Big(\theta-1\Big)\Big(\theta-2\Big)+\alpha^2c^2\kappa^2-6\beta^2c^2\kappa^2\Big(B\theta\Big)\Big(\frac{1}{\theta-1}\Big)$$

$$+\Big(\frac{Q}{6}\Big)^{\frac{\theta-2}{2(\theta-1)}}+\alpha^2c^2\kappa^2\Big(\frac{\theta-1}{3-\theta}\Big)+\frac{\alpha}{3m}-\beta B\theta\Big(\theta-1\Big)+\gamma\sqrt{\frac{Q}{6}}\Big]$$

$$= \frac{1}{2}\Big[1-c^2\kappa^2\Big(\gamma+2\Big)-6c^2\kappa^2\Big(B\theta\Big)^{\frac{2}{\theta-1}}\Big(\theta-1\Big)\Big(\theta-2\Big)+\alpha^2c^2\kappa^2-6\beta^2c^2\kappa^2\Big(B\theta\Big)\Big(\frac{1}{\theta-1}\Big)$$

$$+\Big(\frac{Q}{6}\Big)^{\frac{\theta-2}{2(\theta-1)}}+\alpha^2c^2\kappa^2\Big(\frac{\theta-1}{3-\theta}\Big)+\frac{\alpha}{3m}-\beta B\theta\Big(\theta-1\Big)+\gamma\sqrt{\frac{Q}{6}}\Big]\int\frac{dQ}{Q}$$

$$= c_6\sqrt{Q}+1-c^2\kappa^2\Big(\gamma+2\Big)Q-3c^2\kappa^2\alpha\Big(B\theta\Big)^{\frac{2}{\theta-1}}\Big(\theta-1\Big)\Big(\theta-2\Big)\sqrt{\frac{6}{Q}}+12\alpha^2c^2\kappa^2\Big(\frac{\theta-1}{3-\theta}\Big)$$

$$\Big(\frac{Q}{6}\Big)^{\frac{1}{\theta-1}}+6\beta c^2\kappa^2\Big(B\theta\Big)^{\frac{1}{\theta-1}}\Big(\theta-1\Big)^2\Big(\frac{Q}{6}\Big)^{\frac{\theta-2}{2(\theta-1)}} \tag{3.112}$$

Again,

$$\rho_{de} = -\frac{1}{\kappa^2}\Big[2\dot{H}\Big(1-2f_Q\Big)+\frac{f}{2}+3H^2\Big(1-8f_{QQ}\dot{H}-2f_Q\Big)\Big]$$

$$= \frac{Q}{2}\Big[1-\kappa^2c^2\Big(\gamma+2\Big)\Big(-6c^2\kappa^2\alpha\Big(B\theta\Big)^{\frac{2}{\theta-1}}(\theta-2)\sqrt{\frac{6}{Q}}+\alpha^2c^2\kappa^2\left(\frac{Q}{6}\right)^{\frac{1}{-}6\beta c^2\kappa^2(B\theta)^{\frac{1}{\theta-1}}\theta-1}$$

$$\Big(\frac{Q}{6}\Big)^{\frac{\theta-2}{2(\theta-1)}}\Big)\Big[1-c^2\kappa^2\Big(\gamma+2\Big)-6c^2\kappa^2\Big(B\theta\Big)^{\frac{2}{\theta-1}}\Big(\theta-1\Big)\Big(\theta-2\Big)+\alpha^2c^2\kappa^2-6\beta^2c^2\kappa^2\Big(B\theta\Big)$$

$$\Big(\frac{1}{\theta-1}\Big)+8c_6\sqrt{Q}+1-c^2\kappa^2\Big(\gamma+2\Big)Q-3c^2\kappa^2\alpha\Big(B\theta\Big)^{\frac{2}{\theta-1}}\Big(\theta-1\Big)\Big(\theta-2\Big)\sqrt{\frac{6}{Q}}$$

$$+12\alpha^2c^2\kappa^2\Big(\frac{\theta-1}{3-\theta}\Big)\Big]$$

$$=-\frac{1}{\kappa^2}\Big[2B\theta\Big(\theta-1\Big)+c_6\sqrt{Q}+1-c^2\kappa^2\Big(\gamma+2\Big)Q-3c^2\kappa^2\alpha\Big(B\theta\Big)^{\frac{2}{\theta-1}}\Big(\theta-1\Big)\Big(\theta-2\Big)$$

$$\sqrt{\frac{6}{Q}}+12\alpha^2c^2\kappa^2\Big(\frac{\theta-1}{3-\theta}\Big)+\Big(\frac{Q}{6}\Big)^{\frac{\theta-2}{2(\theta-1)}}+\alpha^2c^2\kappa^2\Big(\frac{\theta-1}{3-\theta}\Big)+\frac{\alpha}{3m}-\beta B\theta\Big(\theta-1\Big)+\gamma\sqrt{\frac{Q}{6}}$$

$$\Big(\frac{1}{\theta-1}\Big)+8c_6\sqrt{Q}+1-c^2\kappa^2\Big(\gamma+2\Big)Q-3c^2\kappa^2\alpha\Big(B\theta\Big)^{\frac{2}{\theta-1}}\Big(\theta-1\Big)\Big(\theta-2\Big)\sqrt{\frac{6}{Q}}\Big] \quad (3.113)$$

$$\rho_{eff}=\frac{1}{f_Q}\Big(\rho+\frac{f}{2}\Big)$$

$$=\frac{1}{f_Q}\Big(\rho_{de}+\frac{f}{2}\Big)$$

$$=c_6\sqrt{Q}+1-c^2\kappa^2\Big(\gamma+2\Big)Q-3c^2\kappa^2\alpha\Big(B\theta\Big)^{\frac{2}{\theta-1}}\Big(\theta-1\Big)\Big(\theta-2\Big)\sqrt{\frac{6}{Q}}+12\alpha^2c^2\kappa^2\Big(\frac{\theta-1}{3-\theta}\Big)$$

$$+1-c^2\kappa^2\Big(\gamma+2\Big)Q-3c^2\kappa^2\alpha\Big(B\theta\Big)^{\frac{2}{\theta-1}}\Big(\theta-1\Big)\Big(\theta-2\Big)+\Big[c^5\frac{1}{2}Q^{\frac{-1}{2}}\Big(1-c^2\kappa^2\Big)^Q$$

$$\Big(\theta-2\Big)\Big(\frac{1}{6}\Big)^{\theta-2}Q^{\theta-3}-c^2\kappa^2\beta B\frac{1}{\sqrt{6}}\Big(\frac{1}{2}Q^{-\frac{1}{2}}ln(Q)+Q^{-\frac{1}{2}}\Big)\Big]^{-1} \quad (3.114)$$

Again,

$$p_{eff}=-2\frac{f(\dot{Q})}{f'(Q)}H+\frac{1}{f'(Q)}\Big(p+\frac{f}{2}\Big)$$

$$=-2\frac{f''(Q)}{f'(Q)}\dot{Q}H+\frac{1}{f'(Q)}\Big(p_{de}+\frac{f}{2}\Big)$$

$$=-2\frac{f''(Q)}{f'(Q)}(12H\dot{H})H+\frac{1}{f'(Q)}\Big(p_{de}+\frac{f}{2}\Big)$$

$$= -24\frac{f''(Q)}{f'(Q)}H^2\dot{H} + \frac{1}{f'(Q)}\Big(p_{de} + \frac{f}{2}\Big)$$

$$= -24\frac{f''(Q)}{f'(Q)}\Big(\frac{Q}{6}\Big)\Big(\frac{-Q}{6n}\Big) + \frac{1}{f'(Q)}\Big(p_{de} + \frac{f}{2}\Big)$$

$$= -24\frac{f''(Q)}{f'(Q)}\Big(\frac{Q}{6}\Big)\Big(\frac{-Q}{6n}\Big) + \frac{1}{f'(Q)}\Big(p_{de} + \frac{f}{2}\Big)$$

$$p_{eff} = \frac{2Q^2}{3n}\frac{f''(Q)}{f'(Q)} + \frac{1}{f'(Q)}\Big(\rho_{de} + \frac{f}{2}\Big)$$

$$= \Big[1 - c^2\kappa^2\Big(\gamma + 2\Big) - 6c^2\kappa^2\Big(B\theta\Big)^{\frac{2}{\theta-1}}\Big(\theta - 1\Big)\Big(\theta - 2\Big) + \alpha^2c^2\kappa^2 - 6\beta^2c^2\kappa^2\Big(B\theta\Big)\Big(\frac{1}{\theta-1}\Big)$$

$$+\frac{1}{2}\Big[1 - c^2\kappa^2\Big(\alpha - \beta\Big) - c^2\kappa^2\beta\Big(B\theta\Big)^{\frac{1}{\theta-1}}\Big(\frac{Q}{6}\Big)^{\frac{\theta-2}{2(\theta-1)}} - c^2\kappa^2\beta B\Big] + 12\alpha^2c^2\kappa^2\Big(\frac{\theta-1}{3-\theta}\Big)$$

$$\Big[2B\theta\Big(\theta - 1\Big) + c_6\sqrt{Q} + 1 - c^2\kappa^2\Big(\gamma + 2\Big)Q - 3c^2\kappa^2\alpha\Big(B\theta\Big)^{\frac{2}{\theta-1}}\Big(\theta - 1\Big)\Big(\theta - 2\Big)$$

$$\sqrt{\frac{6}{Q}} + 12\alpha^2c^2\kappa^2\Big(\frac{\theta-1}{3-\theta}\Big) + \Big(\frac{Q}{6}\Big)^{\frac{\theta-2}{2(\theta-1)}} + \alpha^2c^2\kappa^2\Big(\frac{\theta-1}{3-\theta}\Big) + \frac{\alpha}{3m} - \beta B\theta\Big(\theta - 1\Big) + \gamma\sqrt{\frac{Q}{6}}\Big]^{-1} \tag{3.115}$$

For WEC:

$$\Big[c_6\sqrt{Q}\Big(-3 + \theta\Big) - 2^{3+\frac{1}{1+\theta}}.3^{\frac{-2+\theta}{-1+\theta}}c^2Q^{\frac{1}{-1+\theta}}\alpha\kappa^2 - 2c^2Q\Big(2 + \gamma\Big)\Big(-3 + \theta\Big)\kappa^2 + 2^{3+\frac{1}{1-\theta}}3^{\frac{-2+\theta}{-1+\theta}}$$

$$c^2Q^{\frac{1}{-1+\theta}}\alpha\Big(-1 + \theta\Big)\kappa^2 - 2^{\frac{2+3\theta-\theta^2}{2}}3^{\frac{-(-3+\theta)(-1+\theta)}{2}}\beta\Big(-3 + \theta\Big)\Big(-1 + \theta\Big)^2\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\kappa^2$$

$$6^{\frac{-(-3+\theta)\theta}{2}}c^2Q^{\frac{(-2+\theta)(-1+\theta)}{2}}\beta\Big(-3 + \theta\Big)\Big(-2 + \theta\Big)\Big(-1 + \theta\Big)^3\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\kappa^2 + 9\sqrt{6}c^2\sqrt{\frac{1}{Q}}\alpha$$

$$\Big(-3 + \theta\Big)\Big(-2 + \theta\Big)\Big(-1 + \theta\Big)\Big(B\theta\Big)^{\frac{2}{-1+\theta}}\kappa^2 + 2Q\Big(3 - \theta\Big)1 - c^2\Big(2 + \gamma\Big)\kappa^2\Big[\Big(3 - \theta\Big)$$

$$2 + \frac{c_6}{\sqrt{Q}} - 2C^2\Big(2 + \gamma\Big)\kappa^2 - \frac{2^{3+\frac{1}{1+\theta}}3^{\frac{-2+\theta}{-1+\theta}}c^2Q^{-1+\frac{1}{-1+\theta}}\alpha\kappa^2}{-3+\theta} + 6^{\frac{-(-3+\theta)\theta}{2}}c^2Q^{\frac{-(-3+\theta)\theta}{2}}\beta\Big(-2 + \theta\Big)$$

$$\Big(-1+\theta\Big)^3\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\kappa^2+3\sqrt{6}c^2\Big(\frac{1}{Q}\Big)^{\frac{3}{2}}\alpha\Big(-2+\theta\Big)\Big(-1+\theta\Big)\Big(B\theta\Big)^{\frac{2}{-1+\theta}}\kappa^2\Big]^{-1}\geq 0 \quad (3.116)$$

$$\Big[\Big(1-\theta\Big)2Q\Big(-c_6\sqrt{Q}\Big(-3+\theta\Big)-2Q\Big(-3+\theta\Big)+2^{3+\frac{1}{1-\theta}}3^{\frac{-2+\theta}{-1+\theta}}c^2Q^{\frac{1}{-1+\theta}}c^2Q^{\frac{1}{-1+\theta}}\alpha\kappa^2$$

$$+2c^2Q\Big(2+\gamma\Big)\Big(-3+\theta\Big)\kappa^2-6^{\frac{-(-3+\theta)\theta}{2}}c^2Q^{\frac{(-2+\theta)(-1+\theta)}{2}}\beta\Big(-3+\theta\Big)\Big(-2+\theta\Big)\Big(-1+\theta\Big)^3$$

$$\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\kappa^2-3\sqrt{6}c^2\sqrt{\frac{1}{Q}}\alpha\Big(-3+\theta\Big)\Big(-2+\theta\Big)\Big(-1+\theta\Big)\Big(B\theta\Big)^{\frac{2}{-1+\theta}}\kappa^2-6^{\frac{\theta}{2(-1+\theta)}}B$$

$$\Big(\frac{\sqrt{Q}}{B\theta}\Big)^{\frac{-2+\theta}{-1+\theta}}\theta-\Big(1-6^{\frac{\theta}{2-2\theta}}B\Big(\frac{\sqrt{Q}}{B\theta}\Big)^{\frac{\theta}{-1+\theta}}\Big)\theta c_6\sqrt{Q}\Big(-3+\theta\Big)-2Q\Big(-3+\theta\Big)+2^{3+\frac{1}{1+\theta}}$$

$$3^{\frac{-2+\theta}{-1+\theta}}c^2Q^{\frac{1}{-1+\theta}}\alpha\kappa^2+2c^2Q\Big(2+\gamma\Big)\Big(3+\theta\Big)\kappa^2-6^{\frac{-(-3+\theta)\theta}{2}}c^2Q^{\frac{(-2+\theta)(-1+\theta)}{2}}\beta\Big(-3+\theta\Big)\Big(-2+\theta\Big)$$

$$\Big(-1+\theta\Big)^3\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\kappa^2-3\sqrt{6}c^2\sqrt{\frac{1}{Q}}\alpha\Big(-3+\theta\Big)\Big(-2+\theta\Big)\Big(-1+\theta\Big)\Big(B\theta\Big)^{\frac{2}{-1+\theta}}\kappa^2+3Q$$

$$-c_6\sqrt{Q}\Big(-3+\theta\Big)+2Q\Big(-3+\theta\Big)-2^{3+\frac{1}{1-\theta}}3^{\frac{-2+\theta}{-1+\theta}}c^2Q^{\frac{1}{-1+\theta}}\alpha\kappa^2-2c^2Q\Big(2+\gamma\Big)\Big(-3+\theta\Big)\kappa^2$$

$$-2^{\frac{2+3\theta-\theta^2}{2}}3^{\frac{-(-3+\theta)\theta}{2}}c^2Q^{\frac{(-2+\theta)(-1+\theta)}{2}}\beta\Big(-3+\theta\Big)\Big(-1+\theta\Big)^2\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\kappa^2+6^{\frac{-(-3+\theta)\theta}{2}}c^2$$

$$Q^{\frac{(-2+\theta)(-1+\theta)}{2}}\beta\Big(-3+\theta\Big)\Big(-2+\theta\Big)\Big(-1+\theta\Big)^3\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\kappa^2+9\sqrt{6}c^2\sqrt{\frac{1}{Q}}\alpha\Big(-3+\theta\Big)$$

$$\Big(-2+\theta\Big)\Big(-1+\theta\Big)\Big(B\theta\Big)^{\frac{2}{-1+\theta}}\kappa^2+2Q\Big(3-\theta\Big)\Big(1-c^2\Big(2+\gamma\Big)\kappa^2\Big)\Big]\Big[3Q\Big(-3+\theta\Big)$$

$$\Big(-1+\theta\Big)2+\frac{c_6}{\sqrt{Q}}-2c^2\Big(2+\gamma\Big)\kappa^2-\frac{2^{3+\frac{1}{1-\theta}}3^{\frac{-2+\theta}{-1+\theta}}c^2Q^{-1+\frac{1}{-1+\theta}}\alpha\kappa^2}{-3+\theta}+6^{\frac{-(-3+\theta)\theta}{2}}c^2Q^{\frac{(-3+\theta)\theta}{2}}$$

$$\beta\Big(-2+\theta\Big)\Big(-1+\theta\Big)^3\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\kappa^2+3\sqrt{6}c^2\Big(\frac{1}{Q}\Big)^{\frac{3}{2}}\alpha\Big(-2+\theta\Big)\Big(-1+\theta\Big)\Big(B\theta\Big)^{\frac{2}{-1+\theta}}\kappa^2\Big]\geq 0 \quad (3.117)$$

For NEC:

$$\Big[\Big(1-\theta\Big)2Q\Big(-c_6\sqrt{Q}\Big(-3+\theta\Big)-2Q\Big(-3+\theta\Big)+2^{3+\frac{1}{1-\theta}}3^{\frac{-2+\theta}{-1+\theta}}c^2Q^{\frac{1}{-1+\theta}}c^2Q^{\frac{1}{-1+\theta}}\alpha\kappa^2$$

$$+2c^2Q\Big(2+\gamma\Big)\Big(-3+\theta\Big)\kappa^2-6^{\frac{-(-3+\theta)\theta}{2}}c^2Q^{\frac{(-2+\theta)(-1+\theta)}{2}}\beta\Big(-3+\theta\Big)\Big(-2+\theta\Big)\Big(-1+\theta\Big)^3$$

$$\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\kappa^2-3\sqrt{6}c^2\sqrt{\frac{1}{Q}}\alpha\Big(-3+\theta\Big)\Big(-2+\theta\Big)\Big(-1+\theta\Big)\Big(B\theta\Big)^{\frac{2}{-1+\theta}}\kappa^2-6^{\frac{\theta}{2(-1+\theta)}}B$$

$$\Big(\frac{\sqrt{Q}}{B\theta}\Big)^{\frac{-2+\theta}{-1+\theta}}\theta-\Big(1-6^{\frac{\theta}{2-2\theta}}B\Big(\frac{\sqrt{Q}}{B\theta}\Big)^{\frac{\theta}{-1+\theta}}\Big)\theta c_6\sqrt{Q}\Big(-3+\theta\Big)-2Q\Big(-3+\theta\Big)+2^{3+\frac{1}{1+\theta}}$$

$$3^{\frac{-2+\theta}{-1+\theta}}c^2Q^{\frac{1}{-1+\theta}}\alpha\kappa^2+2c^2Q\Big(2+\gamma\Big)\Big(3+\theta\Big)\kappa^2-6^{\frac{-(-3+\theta)\theta}{2}}c^2Q^{\frac{(-2+\theta)(-1+\theta)}{2}}\beta\Big(-3+\theta\Big)\Big(-2+\theta\Big)$$

$$\Big(-1+\theta\Big)^3\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\kappa^2-3\sqrt{6}c^2\sqrt{\frac{1}{Q}}\alpha\Big(-3+\theta\Big)\Big(-2+\theta\Big)\Big(-1+\theta\Big)\Big(B\theta\Big)^{\frac{2}{-1+\theta}}\kappa^2+3Q$$

$$-c_6\sqrt{Q}\Big(-3+\theta\Big)+2Q\Big(-3+\theta\Big)-2^{3+\frac{1}{1-\theta}}3^{\frac{-2+\theta}{-1+\theta}}c^2Q^{\frac{1}{-1+\theta}}\alpha\kappa^2-2c^2Q\Big(2+\gamma\Big)\Big(-3+\theta\Big)\kappa^2$$

$$-2^{\frac{2+3\theta-\theta^2}{2}}3^{\frac{-(-3+\theta)\theta}{2}}c^2Q^{\frac{(-2+\theta)(-1+\theta)}{2}}\beta\Big(-3+\theta\Big)\Big(-1+\theta\Big)^2\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\kappa^2+6^{\frac{-(-3+\theta)\theta}{2}}c^2$$

$$Q^{\frac{(-2+\theta)(-1+\theta)}{2}}\beta\Big(-3+\theta\Big)\Big(-2+\theta\Big)\Big(-1+\theta\Big)^3\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\kappa^2+9\sqrt{6}c^2\sqrt{\frac{1}{Q}}\alpha\Big(-3+\theta\Big)$$

$$\Big(-2+\theta\Big)\Big(-1+\theta\Big)\Big(B\theta\Big)^{\frac{2}{-1+\theta}}\kappa^2+2Q\Big(3-\theta\Big)\Big(1-c^2\Big(2+\gamma\Big)\kappa^2\Big)\Big]\Big[3Q\Big(-3+\theta\Big)$$

$$\Big(-1+\theta\Big)2+\frac{c_6}{\sqrt{Q}}-2c^2\Big(2+\gamma\Big)\kappa^2-\frac{2^{3+\frac{1}{1-\theta}}3^{\frac{-2+\theta}{-1+\theta}}c^2Q^{-1+\frac{1}{-1+\theta}}\alpha\kappa^2}{-3+\theta}+6^{\frac{-(-3+\theta)\theta}{2}}c^2Q^{\frac{(-3+\theta)\theta}{2}}$$

$$\beta\Big(-2+\theta\Big)\Big(-1+\theta\Big)^3\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\kappa^2+3\sqrt{6}c^2\Big(\frac{1}{Q}\Big)^{\frac{3}{2}}\alpha\Big(-2+\theta\Big)\Big(-1+\theta\Big)\Big(B\theta\Big)^{\frac{2}{-1+\theta}}\kappa^2\Big]\geq 0 \tag{3.118}$$

For DEC:

$$\Big[c_6\sqrt{Q}\Big(-3+\theta\Big)-2^{3+\frac{1}{1-\theta}}3^{\frac{-2+\theta}{-1+\theta}}c^2Q^{\frac{1}{-1+\theta}}\alpha\kappa^2-2c^2Q\Big(2+\gamma\Big)\Big(-3+\theta\Big)\kappa^2+2^{3+\frac{1}{1-\theta}}3^{\frac{-2+\theta}{-1+\theta}}$$

$$c^2Q^{\frac{1}{-1+\theta}}\alpha\Big(-1+\theta\Big)\kappa^2-2^{\frac{2+3\theta-\theta^2}{2}}3^{\frac{-(-3+\theta)\theta}{2}}c^2Q^{\frac{(-2+\theta)(-1+\theta)}{2}}\beta\Big(-3+\theta\Big)\Big(-1+\theta\Big)^2\Big(B\theta\Big)^{\frac{1}{-1+\theta}}$$

$$\kappa^2+6^{\frac{-(-3+\theta)\theta}{2}}c^2Q^{\frac{(-2+\theta)(-1+\theta)}{2}}\beta\Big(-3+\theta\Big)\Big(-2+\theta\Big)\Big(-1+\theta\Big)^3\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\kappa^2+9\sqrt{6}\sqrt{\frac{1}{Q}}\alpha$$

$$\Big(-3+\theta\Big)\Big(-2+\theta\Big)\Big(-1+\theta\Big)^3\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\kappa^2+2Q\Big(3-\theta\Big)1-c^2\Big(2+\gamma\Big)\kappa^2\Big]\Big[\Big(3-\theta\Big)$$

$$2+\frac{c_6}{\sqrt{Q}}-2c^2\Big(2+\gamma\Big)\kappa^2--\frac{2^{3+\frac{1}{1-\theta}}3^{\frac{-2+\theta}{-1+\theta}}c^2Q^{-1+\frac{1}{-1+\theta}}\alpha\kappa^2}{-3+\theta}+6^{\frac{-(-3+\theta)\theta}{2}}c^2Q^{\frac{(-3+\theta)\theta}{2}}\beta\Big(-2+\theta\Big)$$

$$\Big(-1+\theta\Big)^3\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\kappa^2+3\sqrt{6}c^2\Big(\frac{1}{Q}\Big)^{\frac{3}{2}}\alpha\Big(-2+\theta\Big)\Big(-1+\theta\Big)\Big(B\theta\Big)^{\frac{2}{-1+\theta}}\kappa^2\Big]\geq 0 \quad (3.119)$$

$$\Big[c_6\sqrt{Q}\Big(-3+\theta\Big)-2^{3+\frac{1}{1-\theta}}3^{\frac{-2+\theta}{-1+\theta}}c^2Q^{\frac{1}{-1+\theta}}\alpha\kappa^2-2c^2Q\Big(2+\gamma\Big)\Big(-3+\theta\Big)\kappa^2+2^{3+\frac{1}{1-\theta}}3^{\frac{-2+\theta}{-1+\theta}}$$

$$c^2Q^{\frac{1}{-1+\theta}}\alpha\Big(-1+\theta\Big)\kappa^2-2^{\frac{2+3\theta-\theta^2}{2}}3^{\frac{-(-3+\theta)\theta}{2}}c^2Q^{\frac{(-2+\theta)(-1+\theta)}{2}}\beta\Big(-3+\theta\Big)\Big(-1+\theta\Big)^2\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\kappa^2$$

$$+6^{\frac{-(-3+\theta)\theta}{2}}c^2Q^{\frac{(-2+\theta)(-1+\theta)}{2}}\beta\Big(-3+\theta\Big)\Big(-2+\theta\Big)\Big(-1+\theta\Big)^3\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\kappa^2+9\sqrt{6}c^2\sqrt{\frac{1}{Q}}\alpha$$

$$\Big(-3+\theta\Big)\Big(-2+\theta\Big)\Big(-1+\theta\Big)\Big(B\theta\Big)^{\frac{2}{-1+\theta}}\kappa^2+2Q\Big(3-\theta\Big)1-c^2\Big(2+\gamma\Big)\kappa^2\Big]\Big[\Big(3-\theta\Big)$$

$$2+\frac{c_6}{\sqrt{Q}}-2c^2\Big(2+\gamma\Big)\kappa^2--\frac{2^{3+\frac{1}{1-\theta}}3^{\frac{-2+\theta}{-1+\theta}}c^2Q^{-1+\frac{1}{-1+\theta}}\alpha\kappa^2}{-3+\theta}+6^{\frac{-(-3+\theta)\theta}{2}}c^2Q^{\frac{(-3+\theta)\theta}{2}}\beta\Big(-2+\theta\Big)$$

$$\Big(-1+\theta\Big)^3\Big(B\theta\Big)^{\frac{1}{-1+\theta}}\kappa^2+3\sqrt{6}c^2\Big(\frac{1}{Q}\Big)^{\frac{3}{2}}\alpha\Big(-2+\theta\Big)\Big(-1+\theta\Big)\Big(B\theta\Big)^{\frac{2}{-1+\theta}}\kappa^2\Big]^{-1}$$

$$+\frac{1}{3}exp^{-2Re[\frac{ln\frac{\sqrt{Q}}{B\theta}}{-1+\theta}]}|2Q\Big(\frac{\sqrt{Q}}{B\theta}\Big)^{\frac{2}{-1+\theta}}+B\Big(\frac{\sqrt{Q}}{B\theta}\Big)^{\frac{\theta}{-1+\theta}}\theta-6^{\frac{\theta}{2(-1+\theta)}}+6^{\frac{\theta}{2(-1+\theta)}}+B\Big(\frac{\sqrt{Q}}{B\theta}\Big)^{\frac{\theta}{-1+\theta}}\theta|\geq 0 \quad (3.120)$$

For SEC:

$$\Big[2Q\Big(1-\theta\Big)\Big(-c_6\sqrt{Q}\Big(-3+\theta\Big)-2Q\Big(-3+\theta\Big)+2^{3+\frac{1}{1-\theta}}3^{\frac{-2+\theta}{-1+\theta}}c^2Q^{\frac{1}{-1+\theta}}\alpha\kappa^2+2c^2Q$$

$$\Big(2+\gamma\Big)\Big(3+\gamma\Big)\kappa^2-6^{\frac{-(-3+\theta)\theta}{2}}c^2Q^{\frac{(-2+\theta)(-1+\theta)}{2}}\beta\Big(-3+\theta\Big)\Big(-2+\theta\Big)\Big(-1+\theta\Big)^3\Big(B\theta\Big)^{\frac{1}{-1+\theta}}$$

$$\kappa^2-3\sqrt{6}c^2\sqrt{\frac{1}{Q}}\alpha\Big(-3+\theta\Big)\Big(-2+\theta\Big)\Big(-1+\theta\Big)\Big(B\theta\Big)^{\frac{2}{-1+\theta}}\kappa^2\Big)-6^{\frac{\theta}{2(-1+\theta)}}B\Big(1-\theta\Big)\Big]$$

4 Discussion:

4.1 Discussion:

In this thesis, we have analyze the reconstruction schemes of $f(Q)$ gravity in the accelerating universe. Two different models have been discussed here- Granda-Oliveros dark energy model and Chen-Jing dark energy model and have the direct relation with $f(Q)$ gravity. The reconstructed schemes of $f(Q)$ gravity formed in both the models. In the section of Granda-Oliveros we have discussed and proved mathematically, graphically represented it using different model parameters. In the Chen-Jing model we have discussed the reconstruction schemes and different energy condition based on different cases mathematically. As mentioned before, the objective of cosmological reconstruction schemes in accelerating universe to used modified gravities exactly reclaim the features and determine the expansion of our universe.

In order to examine the possibilities of the reconstructed $f(Q)$ model we have examined the energy conditions for the models. Here we have discussed the four types of energy conditions y condition, dominant energy condition, and strong especially the weak energy condition, null energy condition that examine the reconstructed schemes of $f(Q)$ gravity models. Here we have plotted different figures of different cases for GO models i.e, WEC for the GO model in flat and non-flat, NEC for the GO model in flat and non-flat, DEC for the GO model in flat and non-flat, SEC for the GO model in flat

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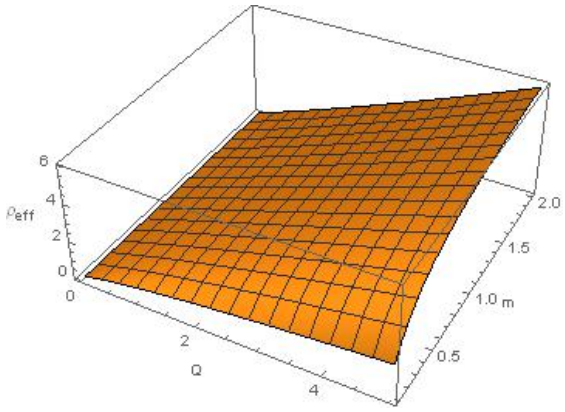

Figure 1: DEC GO MODEL CASE 1

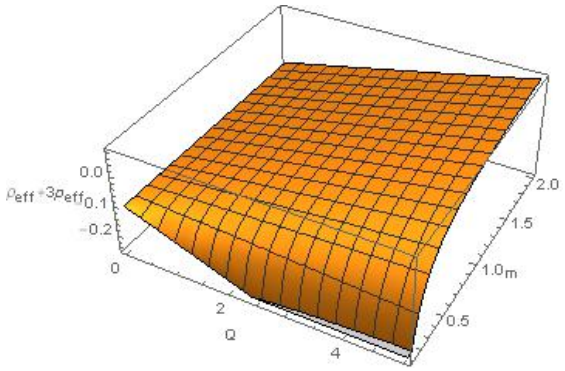

Figure 2: DEC GO MODEL NON-FLAT CASE 1

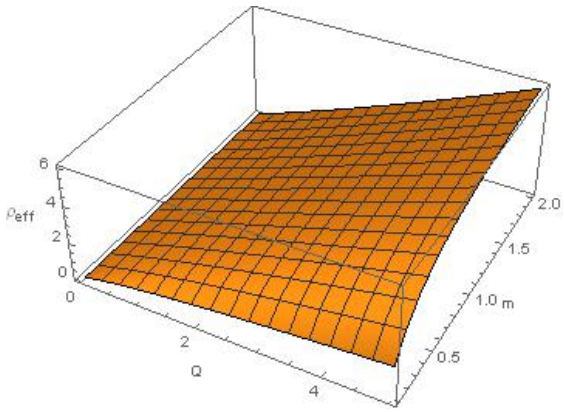

Figure 3: NEC GO MODEL CASE-1

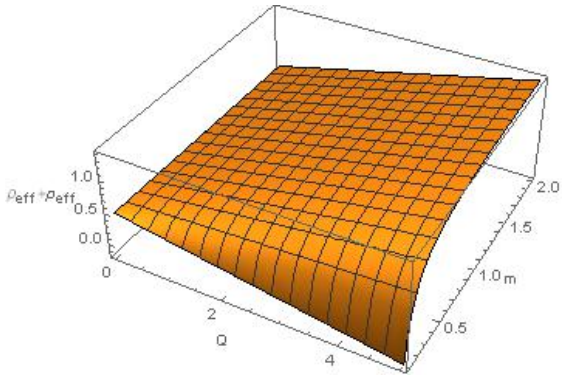

Figure 4: NEC GO MODEL NON-FLAT CASE 1

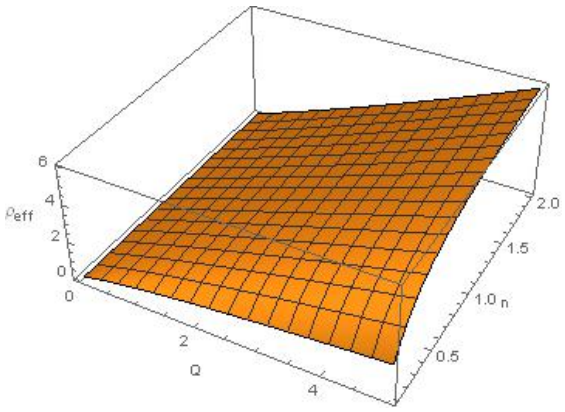

Figure 5: WEC GO MODEL CASE 1

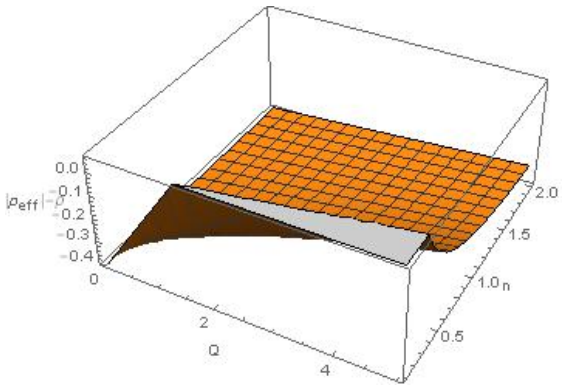

Figure 6: WEC GO MODEL NON-FLAT CASE 1

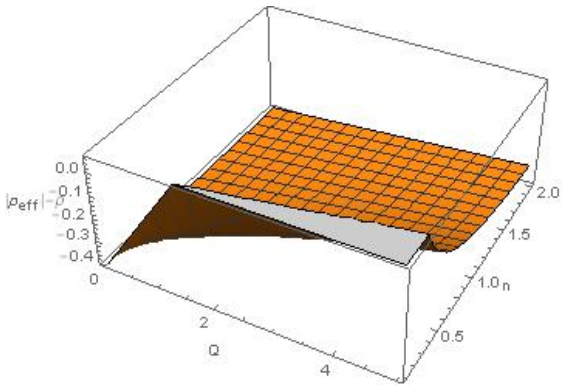

Figure 7: DEC GO CASE II

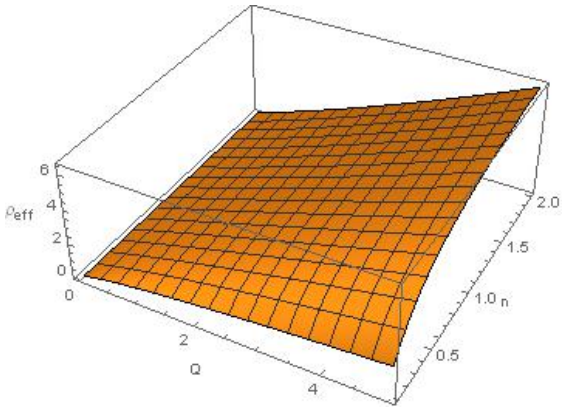

Figure 8: RHO GO CASE II

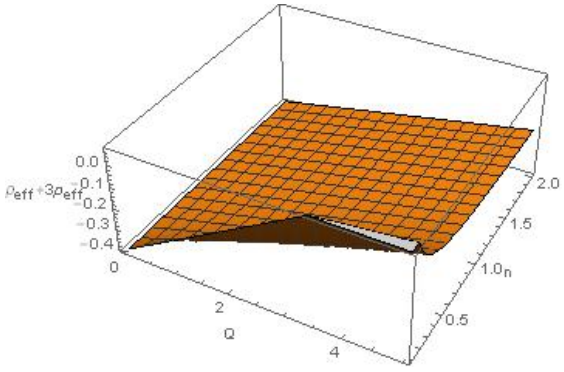

Figure 9: RHO+3p CASE II

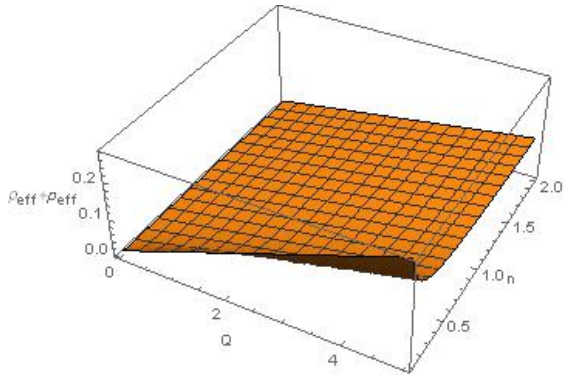

Figure 10: RHO+p GO NON FLAT

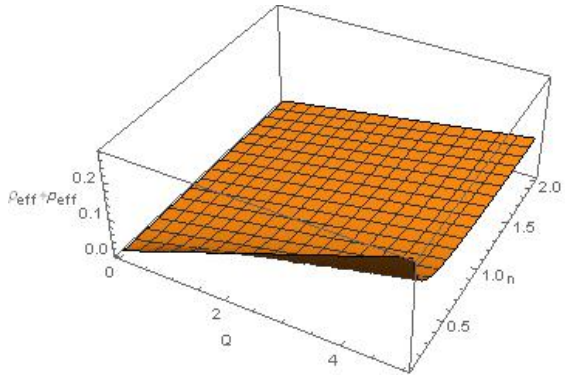

Figure 11: RHO+P GO MODEL CASE II

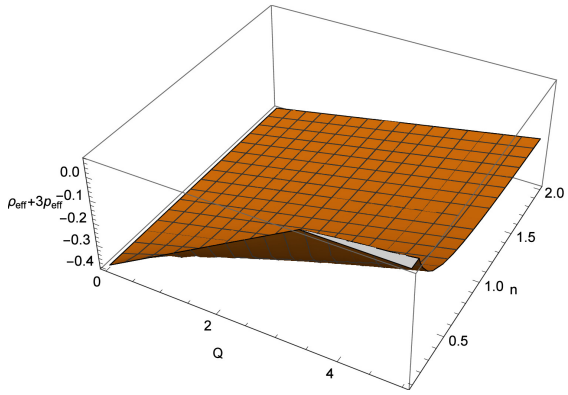

Figure 12: DEC GO MODEL NON-FLAT CASE III

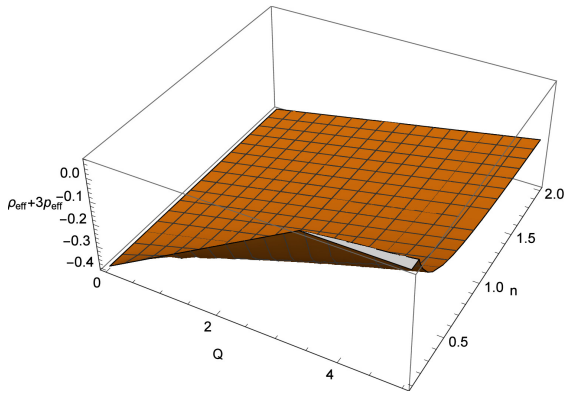

Figure 13: DEC GO MODEL CASE III

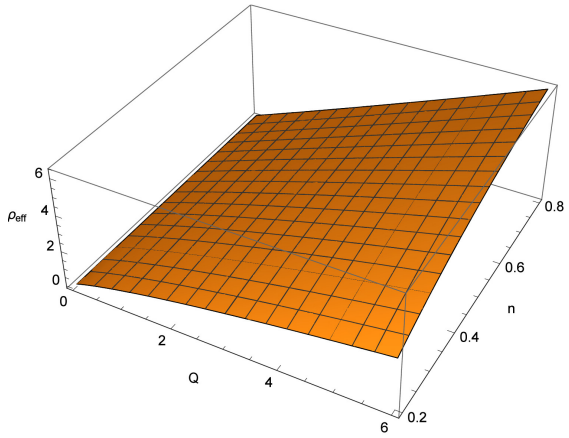

Figure 14: SEC GO MODEL NON-FLAT CASE III

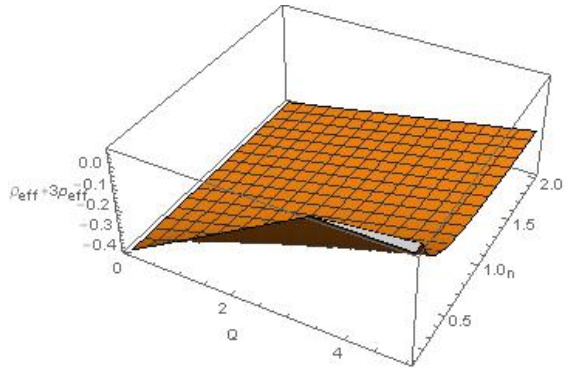

Figure 15: SEC GO MODEL FLAT CASE-3

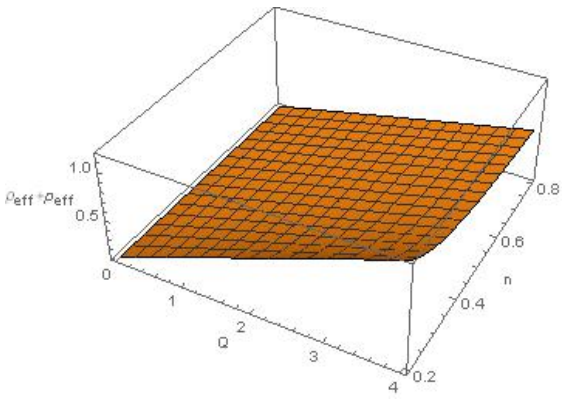

Figure 16: NEC GO MODEL CASE III

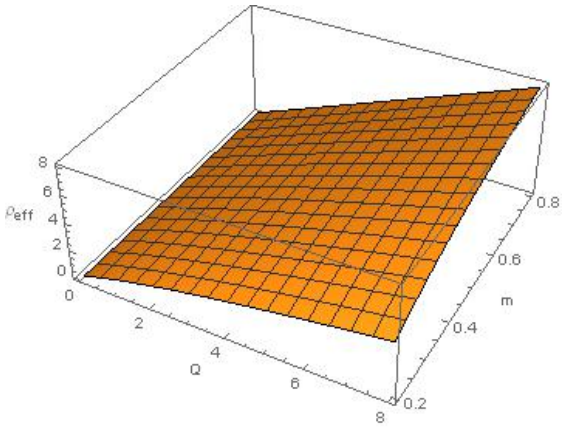

Figure 17: NEC GO MODEL NON-FLAT CASE III

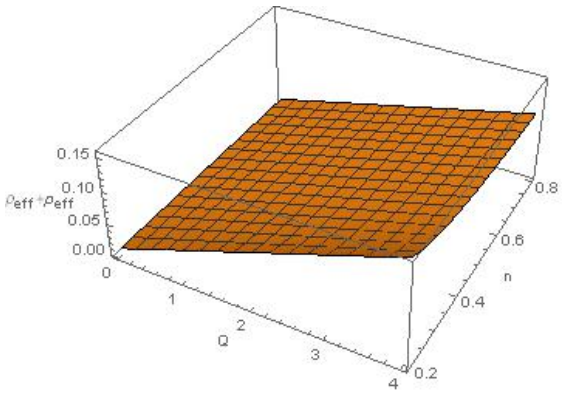

Figure 18: WEC GO MODEL CASE -III

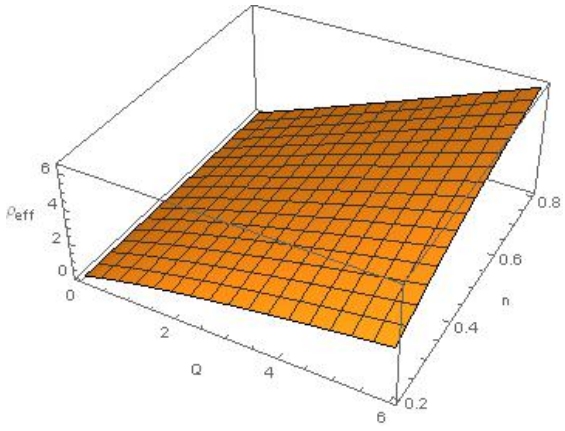

Figure 19: WEC GO MODEL NON-FLAT CASE III

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